The Weil-B.R.S. algebra of a Lie algebra and the anomalous terms in gauge theory

MICHEL DUBOIS-VIOLETTE

Laboratoire de Physique Théorique et Hautes Energies
L.A. 063, Université Paris Sud, Bâtiment 211
91405 Orsay (France)

Abstract. *In this paper, we explain the computation we made in collaboration with M. Talon and C.M. Viallet of anomalous terms in gauge theory* [1], [2], [3]. *We relate our constructions to standard mathematical constructions. The paper is self-contained in the sense that all mathematical concepts and results we use are explained.*

0. INTRODUCTION

The theme of this paper is the work [1] we did in collaboration with M. Talon and C.M. Viallet on the computation of anomalous terms in gauge theory. In [1], we introduced a lot of concepts and methods which are strongly connected with standard mathematical constructions. Since these standard mathematical constructions are not so familiar for physicists, a part of this paper is devoted to explain them and to give examples. We then describe and relate our constructions to these mathematical constructions.

The notion of anomalies comes from the observation that for some invariant classical theories one cannot construct corresponding quantum theories which possess the same invariance. For instance, for non-abelian gauge theory coupled to chiral fermions, the classical (local) action $\Gamma^{(0)}(a, \psi)$ is invariant by gauge transformations while there are obstructions, called anomalies, to the gauge

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invariance of the corresponding quantum action functional $\Gamma(a, \psi)$. Let us remind that $\Gamma(a, \psi)$ is the generating functional for one-particle-irreducible Green «functions», i.e. that it is a functional of classical (test) «fields» (like the classical action $\Gamma^{(0)}(a, \psi)$); this means that in the expression $\Gamma(a, \psi)$, a and ψ may be chosen as smooth and as regular at infinity as needed. The lack of gauge invariance of $\Gamma(a, \psi)$ manifests itself by the non vanishing of the variation $\Delta = \delta\Gamma(a, \psi; \xi)$ of Γ under infinitesimal gauge transformations (ξ are in the Lie algebra of the group of gauge transformations). It turns out that $\delta\Gamma = \Delta$, which is a linear functional in ξ, only depends on a, (and ξ of course), and is local in the sense that one has $\Delta(a; \xi) = \int Q(a; \xi)$, where the integral is taken over the n-dimensional space-time M and where $Q(a; \xi)$ is a n-form on M (which is a functional of a and ξ) such that its value at $x \in M$ only depends on the values at x of a, ξ and a finite number of their derivatives; i.e. $(a, \xi) \to Q(a, \xi)$ is a differential operator which is linear in ξ. By a finite renormalization, Δ is modified by the addition of a term $\delta\Gamma^{\mathrm{loc}}$, where $\Gamma^{\mathrm{loc}}(a) = \int L(a)$ is a local function of a. It follows that the obstruction to invariance is only Δ modulo such $\delta\Gamma^{\mathrm{loc}}$. For a given model Δ is determined by calculation of Feynman graphs, however, these computations are quite cumbersome and it is an objective of the algebraic approach [4], [5], [6] to give the generic form of the obstructions which may occur is any model.

Let G be the structure group of the gauge theory (a finite dimensional compact Lie group) and let $\underline{G}$ be the corresponding group of gauge transformations (i.e. here, the smooth G-valued functions on M). The Lie algebra Lie $(\underline{G})$ of $\underline{G}$ identifies with the smooth functions on M with values in the Lie algebra Lie (G) of G. The infinitesimal right action of $\underline{G}$ on gauge potentials a gives a representation θ of Lie $(\underline{G})$ in the space $\mathcal{P}$ of (polynomial) functionals of a. Thus one may consider the complex $C^*(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ of cochains of Lie $(\underline{G})$ with values in $\mathcal{P}$ (see below in section 1). Let δ denotes the differential of $C^*(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ and consider the subspace $C^*_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G})) = \oplus\, C^k_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ of $C^*(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ where $C^k_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ is the space of the $F(a; \xi_1, \dots, \xi_k)$ of $C^k(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ which are of the form $\int_M Q(a; \xi_1, \dots, \xi_k)$ where $(a, \xi_1, \dots, \xi_k) \to Q(a; \xi_1, \dots, \xi_k)$ is a differential operator with values in the n-form on M. $C^*_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ is stable by δ, so its cohomology $H^*_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ is well defined. Observe that the above $\Delta(a; \xi)$ is in $C^1_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$ and satisfies (by its very definition) the consistency equation of Wess and Zumino [7] $\delta\Delta = 0$ and that finite renormalization modify Δ by the addition of elements of $\delta C^0_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$. Thus the «real anomaly» (i.e. the obstruction to the invariance) lies in $H^1_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}(\underline{G}))$. Similarily by working in $(n-1)$-dimensional fixed-time space, it was pointed out by Faddeev [8], that the obstructions to the elimination of anomalous Schwinger terms $\Delta(a; \xi_1, \xi_2)$ in the equal-time commutation relations of currents are elements of the corresponding $H^2_{\mathrm{loc}}(\mathcal{P}; \mathrm{Lie}\,(\underline{G}))$.

Setting $\Delta = \int Q$ where Q is a differential operator in a and the ξ's, multi-linear antisymmetric in the ξ's, with values in the differential forms, the equation $\delta\Delta = 0$ leads, for Q, to the equation [4] $\delta Q + \mathrm{d}Q' = 0$ for some Q'; d is the exterior differential on differential forms. It turns out that Q' may again be chosen to be a differential operator (of the same type as Q) with values in the differential forms of appropriate degree; this comes from the triviality of the d-cohomology on the appropriate class of differential operators [9], [6]. If $\Delta = \delta \int L$ where L is a differential operator as above, then Q reads $Q = \delta L + \mathrm{d}L'$ for some L' which, for the same reason as before, may be chosen to be again a differential operator of the above type. We say that a Q as above satisfying $\delta Q + \mathrm{d}Q' = 0$ is a *δ-cocycle modulo* d and that if $Q = \delta L + \mathrm{d}L'$ it is a *δ-coboundary modulo* d. Classes of δ-cocycles modulo d up to δ-coboundaries modulo d are the elements of the *δ-cohomology modulo* d. It follows from the above considerations that the relevant cohomology for the problem of anomalous terms in gauge theory is the δ-cohomology modulo d; the local δ-cocycles are obtained by integration of δ-cocycles modulo d on appropriate cycles in space-time.

Let us be slightly more precise. Denote by $\mathcal{C}$ the space of gauge potential on n-dimensional space-time M (with structure group G) and by $\Omega(M) = \oplus\,\Omega^r(M)$ the space of differential forms on M; elements a of $\mathcal{C}$ as well as infinitesimal gauge transformations $\xi \in \mathrm{Lie}\,(\underline{G})$ and differential forms $\omega \in \Omega(M)$ are considered as functions on M. Thus differential operators of $\mathcal{C} \times (\mathrm{Lie}\,(\underline{G}))^s$ in $\Omega^r(M)$ are well defined objects. We denote by $\widetilde{\mathcal{B}}^{r,s}$ the space of differential operators of $\mathcal{C} \times (\mathrm{Lie}\,(\underline{G}))^s$ in $\Omega^r(M)$ which are s-linear antisymmetric in $(\mathrm{Lie}\,(\underline{G}))^s$ and polynomial in a; i.e. if $\omega \in \widetilde{\mathcal{B}}^{r,s}$, $\omega(a; \xi_1, \dots, \xi_s)$ is a r-form on M which depends linearily of each ξ_k, is antisymmetric in the ξ_k's and such that its value at $x \in M$ only depends on the values at x of $a, \xi_1, \dots, \xi_s$ and a finite number of their partial derivatives. We define a product on $\widetilde{\mathcal{B}}^{*,*} = \underset{r,s}{\oplus}\, \widetilde{\mathcal{B}}^{r,s}$, sending $\widetilde{\mathcal{B}}^{r,s} \times \widetilde{\mathcal{B}}^{r',s'}$ in $\widetilde{\mathcal{B}}^{r+r',s+s'}$ by

$$(\omega \cdot \omega')(a; \xi_1, \dots, \xi_s, \dots, \xi_{s+s'}) =$$

$$= \frac{(-1)^{r's}}{(s+s')!} \sum_{\pi \in \mathfrak{S}_{s+s'}} (-1)^{\epsilon(\pi)} \omega(a; \xi_{\pi(1)}, \dots, \xi_{\pi(s)}) \wedge \omega'(a; \xi_{\pi(s+1)}, \dots, \xi_{\pi(s+s')})$$

where $\mathfrak{S}_{s+s'}$ is the group of permutations of $s + s'$ object and $\epsilon(\pi)$ is the signature of $\pi \in \mathfrak{S}_{s+s'}$, $\omega \in \widetilde{\mathcal{B}}^{r,s}$ and $\omega' \in \widetilde{\mathcal{B}}^{r',s'}$. Elements of $\widetilde{\mathcal{B}}^{r,s}$ are said to have *bidegree* (r, s) and *total degree* $r + s$. With the above product $\widetilde{\mathcal{B}}^{*,*}$ is a *graded-commutative algebra* for the graduation corresponding to the total degree (see below in 1.1

for the definition of graded-commutative algebras). On $\tilde{B}^{*,*}$ one defines two differentials d and δ by $(\mathrm{d}\omega)(a; \xi_1, \dots, \xi_s) = \mathrm{d}(\omega(a; \xi_1, \dots, \xi_s))$, where $\omega \in \tilde{B}^{*,s}$ and d is the exterior differential in the left hand side, and by $(\delta\omega)(a; \xi_0, \dots, \xi_1, \dots, \xi_s) = \sum_{0 \leqslant k \leqslant s} (-1)^{k+r} \theta(\xi_k) \omega(a; \xi_0, \dots, \hat{\xi}_k, \dots, \xi_s) + \sum_{0 \leqslant l < m \leqslant s} (-1)^{l+m+r} \omega(a; [\xi_l, \xi_m], \xi_0, \dots, \hat{\xi}_l, \dots, \hat{\xi}_m, \dots, \xi_s)$, where $\omega \in \tilde{B}^{r,s}$, $\hat{}$ stands for omission and $\theta(\xi)$ is induced by the (infinitesimal) right action $a \mapsto \mathrm{d}\xi + [a, \xi]$ of $\xi \in \mathrm{Lie}(\underline{G})$. One has $\mathrm{d}^2 = 0$, $\delta^2 = 0$, $\mathrm{d}\delta + \delta\mathrm{d} = 0$

$$\mathrm{d}\tilde{B}^{r,s} \subset \tilde{B}^{r+1,s} \qquad \text{and} \qquad \delta\tilde{B}^{r,s} \subset \tilde{B}^{r,s+1}$$

d, δ and $\mathrm{d} + \delta$ are three differentials on $\tilde{B}^{*,*}$, d and δ are homogeneous for the bidegree, d is of bidegree $(1, 0)$ and δ is of bidegree $(0, 1)$, while $\mathrm{d} + \delta$ is only homogeneous for the total degree (and of degree one of course). $\mathrm{d}\tilde{B}^{*,*}$ is stable by δ so $\mathrm{d}\tilde{B}^{*,*}$ and $\tilde{B}^{*,*}/\mathrm{d}\tilde{B}^{*,*}$ are complexes for δ and the δ-cohomology modulo d on which we are interested for the problem of anomalous terms in gauge theory is just the cohomology of $\tilde{B}^{*,*}/\mathrm{d}\tilde{B}^{*,*}$; it is a bigraded space. Notice that one has the short exact sequence of (bigraded) δ-complexes

$$0 \to \mathrm{d}\tilde{B}^{*,*} \subsetneq \tilde{B}^{*,*} \xrightarrow{p} \tilde{B}^{*,*}/\mathrm{d}\tilde{B}^{*,*} \to 0$$

and that, one the other hand, one has the exact sequence

$$0 \to \tilde{Z}^{*,*}(\mathrm{d}) \subsetneq \tilde{B}^{*,*} \xrightarrow{\mathrm{d}} \mathrm{d}\tilde{B}^{*,*} \to 0$$

and

$$0 \to \mathrm{d}\tilde{B}^{*,*} \subsetneq \tilde{Z}^{*,*}(\mathrm{d}) \to \tilde{H}^{*,*}(\mathrm{d}) \to 0,$$

where $\tilde{H}^{*,*}(\mathrm{d})$ is the d-cohomology of $\tilde{B}^{*,*}$ and $\tilde{Z}^{*,*}(\mathrm{d})$ is the space of d-cocycles of $\tilde{B}^{*,*}$. Thus triviality of the d-cohomology, i.e. $\tilde{H}^{*,*}_+(\mathrm{d}) = 0$, implies that we have in *positive degrees* a short exact sequence of δ-complexes

$$0 \to (\tilde{B}^{*,*}/\mathrm{d}\tilde{B}^{*,*})_+ \xrightarrow{i} \tilde{B}^{*,*}_+ \xrightarrow{p} (\tilde{B}^{*,*}/\mathrm{d}\tilde{B}^{*,*})_+ \to 0$$

where i is induced by $\mathrm{d} : \tilde{B}^{*,*} \to \tilde{B}^{*,*}$. From this, we obtain, in cohomology an exact triangle relating the δ-cohomology modulo d and the δ-cohomology in positive degrees

$$\begin{tikzcd} & \tilde{H}^{*,*}_+(\delta) \arrow[dl, "p^{\#}"'] & \\ \tilde{H}^{*,*}_+(\delta, \bmod(\mathrm{d})) \arrow[rr, "\partial"] & & \tilde{H}^{*,*}_+(\delta, \bmod(\mathrm{d})) \arrow[ul, "i^{\#}"'] \end{tikzcd}$$

Actually we shall work in an algebra $B^{*,*}$ which is smaller that $\tilde{B}^{*,*}$; this will be sufficient to work out all the known examples and is natural in connection with the index theorem. Let (E_α) be a basis of Lie (G) and let us introduce $A^\alpha : \mathcal{C} \to \Omega^1(M)$ and $\chi^\alpha : \text{Lie}\,(\underline{G}) \to \Omega^0(M)$ by $A^\alpha(a) = a^\alpha$ (remembering that a gauge potential is a one-form with values in Lie (G)) and by $\chi^\alpha(\xi) = \xi^\alpha$ (remembering that an element of Lie $(\underline{G})$ is a function with values in Lie (G)). By definition the A^α's are elements of $\tilde{\mathcal{B}}^{1,0}$ and the χ^α are elements of $\tilde{\mathcal{B}}^{0,1}$. We define $\mathcal{B}^{*,*}$ to be the smallest subalgebra of $\tilde{\mathcal{B}}^{*,*}$ which is stable by d and by δ and contains the A^α's and the χ^α's; i.e. it is the subalgebra of $\tilde{\mathcal{B}}^{*,*}$ generated by the A^α, $\mathrm{d}A^\alpha$, δA^α, χ^α, $\mathrm{d}\chi^\alpha$ and $\delta\chi^\alpha$. Of course the elements of $\mathcal{B}^{*,*}$ are very special types of differential operators, for instance they are of first order at most, and it would be nice to be able to compute $\tilde{H}^{*,*}$ $(\delta, \mathrm{mod}\,(\mathrm{d}))$; (the $\tilde{H}^{k,0}$ $(\delta\ \mathrm{mod}\,(\mathrm{d}))$ contain more elements than the ones coming from $\mathcal{B}^{k,0}$, the Yang-Mills lagrangian for instance, but one may expect that it is essentially all what is lost by working with $\mathcal{B}^{*,*}$ instead of $\tilde{\mathcal{B}}^{*,*}$).

The action of d and δ on the A^α and the χ^α is conveniently described by introducing the following elements of Lie $(G) \otimes \mathcal{B}^{*,*}$: $A = E_\alpha \otimes A^\alpha$, $\mathrm{d}A = E_\alpha \otimes \mathrm{d}A^\alpha$, $\delta A = E_\alpha \otimes \delta A^\alpha$, $\chi = E_\alpha \otimes \chi^\alpha$, $\mathrm{d}\chi = E_\alpha \otimes \mathrm{d}\chi^\alpha$ and $\delta\chi = E_\alpha \otimes \delta\chi^\alpha$. Then one has

$$(*) \qquad \mathrm{d}A + \frac{1}{2}[A, A] = (\mathrm{d} + \delta)(A + \chi) + \frac{1}{2}[A + \chi, A + \chi]$$

where the bracket in Lie $(G) \otimes \mathcal{B}^{*,*}$ is defined by $[X \otimes \omega, X' \otimes \omega'] = [X, X'] \otimes \omega\omega'$ for $X, X' \in$ Lie (G) and $\omega, \omega' \in \mathcal{B}^{*,*}$. Notice that $F = \mathrm{d}A + \frac{1}{2}[A, A] = E_\alpha \otimes F^\alpha$ with $F^\alpha : \mathcal{C} \to \Omega^2(M)$ is given by $F^\alpha(a) = \mathfrak{f}^\alpha(a)$ where $\mathfrak{f}^\alpha(a)$ is the «field-strength» of a. More generally, given a Lie algebra $\mathfrak{g}$, any bigraded algebra $\mathcal{B}^{*,*}$, with differentials d and δ, equipped with an element $A + \chi$ of $\mathfrak{g} \otimes \tilde{\mathcal{B}}^{*,*}$ of degree one (A of bidegree $(1, 0)$ and χ of bidegree $(0, 1)$ such that the equality $(*)$ holds, was called in [1] a *B.R.S. algebra over* $\mathfrak{g}$. Thus both the above $\tilde{\mathcal{B}}^{*,*}$ and $\mathcal{B}^{*,*}$ are B.R.S. algebras over Lie (G).

One has an obvious natural notion of homorphism of B.R.S. algebras over $\mathfrak{g}$ and it turns out that, in this category of B.R.S. algebras over $\mathfrak{g}$ there is a universal initial object $\mathcal{A}(\mathfrak{g})$ which was called in [1] *the universal B.R.S. algebra of the Lie algebra* $\mathfrak{g}$ and which we now call *The Weil-B.R.S. algebra of the Lie algebra* $\mathfrak{g}$; this latter terminology, suggested by R. Stora, comes from the following observations. On $\mathcal{A}(\mathfrak{g})$ and on most B.R.S. algebras over $\mathfrak{g}$ of interest there is a natural *operation of the Lie algebra* $\mathfrak{g}$ in the sense of H. Cartan [10] (see the definitions in 1.3, 1.4) so we also have the notion of a *B.R.S. operation*; this is just a bigraded Cartan operation with an algebraic connection satisfying the above identity $(*)$. Then $\mathcal{A}(\mathfrak{g})$ plays the same role with respect to B.R.S. $\mathfrak{g}$-operations as the one of the Weil algebra $\mathcal{W}(\mathfrak{g})$ with respect to $\mathfrak{g}$-operations with connections (again see the sections 1 and 2 for the definitions and examples). In fact $\mathcal{A}(\mathfrak{g})$ contains $\mathcal{W}(\mathfrak{g})$ (and even a one-parameter family $\mathcal{W}_t(\mathfrak{g})$ of Weil algebra) and is just what is needed to generalize Cartan maps (by the «descent equations») and transgressions. This generalized transgression leads to the computation of the δ-cohomology modulo d of $\mathcal{A}(\mathfrak{g})$, [1] which we shall describe.

In the case of $\mathcal{B}^{*,*}$, the canonical homorphism of $\mathcal{A}(\mathfrak{g})$ in $\mathcal{B}^{*,*}$ is surjective and induces isomorphisms of vector spaces in bidegrees (r, s) such that $r \leqslant n = \dim(M)$, [1]; it follows that the δ-cohomology modulo d of $\mathcal{B}^{*,*}$ is completely known from the one of $\mathcal{A}(\mathfrak{g})$.

As it is well known gauge potentials are connections on the trivial G-principal bundle $M \times G$ seen in the section $x \mapsto (x, 1)$ corresponding to the trivialisation. For a non trivial G-principal bundle P over M one may generalize, (and this will be done below), the constructions and, as explained in the reference [20], one has a similar correspondence between the local cohomology of the Lie algebra of the group of gauge transformations (infinitesimal automorphisms of P) and the δ-cohomology modulo d by introducing a reference connection a_0 on P. In the case of the trivial bundle one chooses implicitely a_0 to be the flat connection corresponding to the trivialisation in order to identify the affine space $\mathcal{C}$ of connections with the vector space of differential one-form on M with values in Lie (G).

Our notations are standard except that here an associative algebra is assumed to have a unit, (generically denoted by $\mathbb{1}$), except otherwise stated; for instance when we speak of a subalgebra generated by some elements, it means the smallest subalgebra with unit which contains these elements.

A very complete reference for the notions of operations, Weil algebras and the cohomology of Lie algebras is [11]; for homology, beside [11], all what we use is described, for instance in [12], [13].

1. CARTAN OPERATIONS, WEIL ALGEBRAS AND TRANSGRESSIONS

1.1. Graded commutative differential algebras

All vector spaces, algebras etc... considered here are on the field $\mathbb{R}$ or on the field $\mathbb{C}$. A *graded algebra* will be an associative algebra $\mathscr{A}$ equipped with a graduation over the integers $\mathbb{N}$, $\mathscr{A} = \underset{n \in \mathbb{N}}{\oplus} \mathscr{A}^n$, such that the product satisfies $\mathscr{A}^m . \mathscr{A}^n \subset \mathscr{A}^{m+n}$. The elements of $\mathscr{A}^n$ are called *homogeneous elements of $\mathscr{A}$ of degree n*. A linear mapping L of $\mathscr{A}$ in itself is said to be *homogeneous of degree k*, $(k \in \mathbb{Z})$, if $L(\mathscr{A}^n) \subset \mathscr{A}^{n+k}$ for any $n \in \mathbb{N}$. A *derivation* of $\mathscr{A}$ is a homogeneous linear endomorphism θ of $\mathscr{A}$ of even degree satisfying $\theta(xy) = (x)y + x\theta(y)$, $\forall x, y \in \mathscr{A}$. An *antiderivation* of $\mathscr{A}$ is a homogeneous linear endomorphism δ of $\mathscr{A}$ of odd degree satisfying $\delta(x . y) = \delta(x)y + (-1)^n \times \delta(y)$,

$\forall x \in \mathscr{A}^n$, $\forall y \in \mathscr{A}$. Derivations and antiderivations are called graded-derivations of $\mathscr{A}$; they form a $\mathbb{Z}$-graded Lie algebra for the graded commutator. Indeed if θ_1 and θ_2 are two derivations and if δ_1 and δ_2 are two antiderivations, then $\theta_1 \theta_2 - \theta_2 \theta_1$ and $\delta_1 \delta_2 + \delta_2 \delta_1$ are derivations and $\theta_i \delta_j - \delta_j \theta_i$ are anti-derivations.

A graded algebra $\mathscr{A}$ is said to be *graded commutative* or to be a graded commutative algebra if for any $\alpha_m \in \mathscr{A}^m$ and $\beta_n \in \mathscr{A}^n$ one has $\alpha_m \cdot \beta_n = (-1)^{mn} \beta_n \cdot \alpha_m$, $(n, m \in \mathbb{N})$.

A *differential* on a graded algebra $\mathscr{A}$ is an antiderivation of degree one d of $\mathscr{A}$ satisfying $\mathrm{d}^2 = 0$ and a graded algebra equipped with a differential is called a *graded differential algebra*. If $(\mathscr{A}, \mathrm{d})$ is a graded differential algebra, an element A of $\mathscr{A}$ is called a *cocycle* if $\mathrm{d}A = 0$ (i.e. $A \in \ker(\mathrm{d})$); if $A \in \mathrm{d}\,\mathscr{A}$ then A is a *coboundary* of $\mathscr{A}$; coboundaries are, of course cocycles in view of $\mathrm{d}^2 = 0$. The set $Z(\mathscr{A})$ of all cocycles of $\mathscr{A}$ is a graded subalgebra of $\mathscr{A}$, $(Z(\mathscr{A}) = \bigoplus_{n \in \mathbb{N}} Z^n(\mathscr{A})$, $Z^n(\mathscr{A}) = Z(\mathscr{A}) \cap \mathscr{A}^n)$, and the set $B(\mathscr{A})$ of all coboundaries of $\mathscr{A}$ is a two-sided graded ideal of $Z(\mathscr{A})$. It follows that $H(\mathscr{A}) = Z(\mathscr{A})/B(\mathscr{A})$ is a graded algebra which is called the *cohomology algebra* of $\mathscr{A}$; $H(\mathscr{A}) = \bigoplus_{n \in \mathbb{N}} H^n(\mathscr{A})$ where $H^n(\mathscr{A}) = Z^n(\mathscr{A})/B^n(\mathscr{A})$ is the *n-th cohomology space of* $\mathscr{A}$. If $\mathscr{A}$ is a graded commutative differential algebra, then $H(\mathscr{A})$ is a graded commutative algebra.

Given two graded algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, the vector space $\mathscr{A}_1 \otimes \mathscr{A}_2$ becomes a graded algebra if we define the graduation by $(\mathscr{A}_1 \otimes \mathscr{A}_2)^n = \bigoplus_{m=0}^{m=n} \mathscr{A}_1^m \otimes \mathscr{A}_2^{n-m}$ and the product by $(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (-1)^{mn} x_1 y_1 \otimes x_2 y_2$ for $x_2 \in \mathscr{A}_2^m$, $y_1 \in \mathscr{A}_1^n$, $x_1 \in \mathscr{A}_1$ and $y_2 \in \mathscr{A}_2$; $\mathscr{A}_1 \otimes \mathscr{A}_2$ is *the tensor product of the graded algebras* $\mathscr{A}_1$ *and* $\mathscr{A}_2$. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are graded commutative then $\mathscr{A}_1 \otimes \mathscr{A}_2$ is also graded commutative and this is the very reason for the appearance of the $(-1)^{mn}$ in the definition of the product. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are graded differential algebras with differential d_1 and d_2 then $\mathscr{A}_1 \otimes \mathscr{A}_2$ becomes a graded differential algebra if we define its differential d by $\mathrm{d}(x_1 \otimes x_2) = \mathrm{d}_1 x_1 \otimes x_2 + (-1)^n x_1 \otimes \mathrm{d}_2 x_2$ for $x_1 \in \mathscr{A}_1^n$ and $x_2 \in \mathscr{A}_2$; Furthermore one has $H(\mathscr{A}_1 \otimes \mathscr{A}_2) = H(\mathscr{A}_1) \otimes H(\mathscr{A}_2)$.

Notice that if $\mathscr{A}$ is generated by $\mathscr{A}^0$ and $\mathscr{A}^1$, then a derivation or an antiderivation of $\mathscr{A}$ is fixed when its values on $\mathscr{A}^0$ and $\mathscr{A}^1$ are known.

In the rest of this lecture, we shall be interested only in graded commutative differential algebras (although many of the following concepts make sense also in the graded non-commutative case).

An example of graded commutative differential algebra is the de Rham complex of exterior differential forms $\Omega(M)$ on a manifold M, the corresponding cohomology algebra $H(\Omega(M)) = H^*(M)$ is *the de Rham cohomology of M.*

It is well known and easy to check that dim $H^0(\Omega(M))$ is the number of connected components of M, in particular $H^0(\Omega(M)) = \mathbb{R}$ (or $\mathbb{C}$ if one considers complex valued forms) if and only if M is connected; this is the origin of the following terminology. A graded commutative algebra $\mathscr{A}$ is said to be *connected* (or connected in degree zero) if $\mathscr{A}^0$ is the ground field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), i.e. $\mathscr{A} = \mathbb{K} \oplus \oplus \mathscr{A}^+$, where $\mathscr{A}^+$ denotes $\oplus_{n \geqslant 1} \mathscr{A}^n$.

A connected graded commutative algebra is said to be *free* if there is a finite set of homogeneous elements $\{e_\alpha\}$ of $\mathscr{A}^+$ which are free of algebraic relations beside graded commutativity and which generate $\mathscr{A}^+$, (i.e. any element of $\mathscr{A}^+$ is a linear combination of products of the e_α's).

Let us say a few words on the structure of the free connected graded commutative differential algebras. An example of such an algebra is the free connected graded commutative algebra $\mathscr{C}(x, \mathrm{d}x)$ generated by an element x of degree $n \geqslant 1$ and its differential $\mathrm{d}x$. If n is odd $x^2 = 0$ so a basis of $\mathscr{C}(x, \mathrm{d}x)$ is 1, x, $\mathrm{d}x$, $x\mathrm{d}x$, $\mathrm{d}x^2, \dots, (\mathrm{d}x)^n, x(\mathrm{d}x)^n, \dots$ and if n is even $(\mathrm{d}x)^2 = 0$ so a basis of $\mathscr{C}(x, \mathrm{d}x)$ is 1, x, $\mathrm{d}x$, x^2, $x\mathrm{d}x, \dots, x^{n+1}$, $x^n\,\mathrm{d}x, \dots$ therefore, in any case all cocycles in $\mathscr{C}^+(x, \mathrm{d}x)$ are coboundaries so $H(\mathscr{C}(x, \mathrm{d}x)) = H^0(\mathscr{C}(x, \mathrm{d}x)) = \mathbb{K}$. By definition a *contractible* [14] *(differential) algebra* is a tensor product $C = \mathscr{C}(x_1, \mathrm{d}x_1) \otimes \dots \otimes \mathscr{C}(x_p, \mathrm{d}x_p)$ of algebras of the above type; for such an algebra we have $H^n(C) = 0$ for $n \geqslant 1$ and of course $H^0(C) = \mathbb{K}$ (the ground field). Another prototype of free connected graded commutative differential algebra is a differential algebra $\mathscr{M}$ which is free connected graded commutative and such that $\mathrm{d}\mathscr{M} \subset \mathscr{M}^+ \cdot \mathscr{M}^+$; such a differential algebra is called *minimal*. One has the following theorem:

THEOREM 1. *Every free connected graded commutative differential algebra is isomorphic to the tensor product of a unique minimal algebra and a unique contractible algebra* [14]. ∎

Later on, we shall have to compute the cohomologies of such algebras and the strategy will be to throw away the contractible parts and to identify the minimal parts with «known objects».

1.2. Cohomology of Lie algebras

a) *Finite dimensional Lie algebras.* The structure of the contractible algebras is completely apparent from their definition; let us now describe the simplest non-trivial minimal algebras [14], namely the ones which are generated in degree one. Such an algebra $\mathscr{M}$ being graded commutative connected and free is necessarily the exterior algebra $\Lambda\mathscr{M}^1$ over the finite dimensional space $\mathscr{M}^1$ of elements of degree one of $\mathscr{M}$, (i.e. $\mathscr{M} = \Lambda\mathscr{M}^1$) and since it is minimal we have $\mathrm{d}\mathscr{M}^1 \subset$

$\subset \mathscr{M}^1 \cdot \mathscr{M}^1 = \Lambda^2 \mathscr{M}^1$. Let $\mathfrak{g}$ be the dual space of $\mathscr{M}^1$, so $\mathscr{M}^1 = \mathfrak{g}^* = ((\mathscr{M}^1)^*)^*$ (since these spaces are finite dimensional). By transposition of the linear map $\mathrm{d} : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$, one defines an antisymmetric bilinear bracket on $\mathfrak{g}$ by writing $\mathrm{d}\omega(X, Y) = -\omega([X, Y])$ for any $X, Y \in \mathfrak{g}$ and any $\omega \in \mathfrak{g}^* = \mathscr{M}^1$. Then, one easily verifies that $\mathrm{d}^2 = 0$ on $\Lambda \mathfrak{g}^*$ (together with the antiderivation property of d) is equivalent to the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, X, Y, Z \in \mathfrak{g}.$$

Thus any minimal algebra which is generated in degree one is the exterior algebra over the dual space $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. Conversely let $\mathfrak{g}$ be a finite dimensional Lie algebra, define $\mathrm{d} : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$ by (the co-bracket) $\mathrm{d}\omega(X, Y) = -\omega([X, Y])$ and extend it as antiderivation of the graded commutative algebra $\Lambda\mathfrak{g}^*$; then the Jacobi identity implies $\mathrm{d}^2 = 0$ on $\mathfrak{g}^*$ and therefore $\mathrm{d}^2 = 0$ on $\Lambda\mathfrak{g}^*$ (since d^2 is a derivation vanishing on the generators). The differential algebra $\Lambda\mathfrak{g}^*$ so defined is obviously minimal and generated in degree one. By definition *the cohomology $H^*(\mathfrak{g})$ of the (finite dimensional) Lie algebra* $\mathfrak{g}$ is the cohomology of the differential algebra $\Lambda\mathfrak{g}^*$.

b) *Relation with the de Rham cohomology of Lie groups*. Let us now assume that $\mathfrak{g}$ is the Lie algebra Lie (G) of a connected finite dimensional Lie group G. Then, $\mathfrak{g}$ identifies (by left translation) as the left-invariant vector fields on G so $\Lambda\mathfrak{g}^*$ identifies with the algebra of left-invariant differential forms on G. It is easy to show that this algebra is stable by exterior differentiation and that the exterior differential coincides there with the above defined (in a)) differential on $\Lambda\mathfrak{g}^*$. Therefore $\Lambda\mathfrak{g}^*$ identifies with a differential subalgebra of the algebra $\Omega(G)$ of differential forms on G. This inclusion induces in cohomology an homomorphism $i^\# : H^*(\mathfrak{g}) \to H^*(G)$ of the cohomology of $\mathfrak{g} =$ Lie (G) in the de Rham cohomology of G which is known to be an isomorphism when G is compact [15]. Thus for a connected compact finite dimensional Lie group G, H^*(Lie (G)) is the de Rham cohomology of G.

c) *Co-adjoint actions and the case of reductive Lie algebras*. Let $\mathfrak{g}$ be again a finite dimensional Lie algebra. $\mathfrak{g}$ acts on itself by the *adjoint representation*; by duality there is a corresponding action of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ called the *co-adjoint representation* which extends to $\Lambda\mathfrak{g}^*$ as a Lie-algebra homomorphism $X \to L_X$ $(X \in \mathfrak{g})$ from $\mathfrak{g}$ in the Lie-algebra of derivations (of degree zero) of $\Lambda\mathfrak{g}^*$. There is another way to describe the situation which will become of importance in the next paragraph which is the following. For any $X \in \mathfrak{g}$ let i_X be the unique antiderivation of $\Lambda\mathfrak{g}^*$ such that $i_X\omega = \omega(X)$ for $\omega \in \mathfrak{g}^*$ $(= \Lambda^1\mathfrak{g}^*)$ and define the derivation of degree zero L_X of $\Lambda\mathfrak{g}^*$ by $L_X = \mathrm{d}i_X + i_X\mathrm{d}$, (1) we have;

(2) $$L_{[X,Y]} = L_X L_Y - L_Y L_X$$

(3) $$L_X i_Y - i_Y L_X = i_{[X,Y]}$$

for any X, $Y \in \mathfrak{g}$. Furthermore, $(L_X \omega)(Y) = \omega([Y, X])$ for $\omega \in \mathfrak{g}^*$, so L_X induces the co-adjoint representation on $\mathfrak{g}^*$ and therefore coincides with the previously defined L_X. One has $L_X \mathrm{d} = \mathrm{d} L_X$, $\forall X \in \mathfrak{g}$. Let e_α be a basis of $\mathfrak{g}$ with dual basis e^α of $\mathfrak{g}^*$ (i.e. $e^\alpha(e_\beta) = \delta^\alpha{}_\beta$), then one easily verifies the *Koszul formula* [16]

(4) $$\mathrm{d}\omega = \frac{1}{2} \sum_\alpha e^\alpha \wedge L_{e_\alpha}(\omega), \qquad \forall \omega \in \Lambda \mathfrak{g}^*.$$

One *calls invariant form on* $\mathfrak{g}$ an element $\omega \in \Lambda \mathfrak{g}^*$ such that $L_X \omega = 0$ for any $X \in \mathfrak{g}$. The space $\mathscr{T}_\Lambda(\mathfrak{g})$ of invariant forms on $\mathfrak{g}$ is a graded subalgebra of $\Lambda \mathfrak{g}^*$ which, in view of the above Koszul formula, consists of cocycles; therefore we have a canonical homomorphism $\mathscr{T}_\Lambda(\mathfrak{g}) \to H^*(\mathfrak{g})$. Notice that when $\mathfrak{g} = \mathrm{Lie}\,(G)$ (with G connected) and $\Lambda \mathfrak{g}^*$ is identified with left-invariant forms on G (as in b)), then $\mathscr{T}_\Lambda(\mathfrak{g})$ identifies with bi-invariant forms on G (i.e. invariant by left and right translations).

By definition, *a reductive Lie algebra* is a Lie algebra which is the direct product of a semi-simple Lie algebra and an abelian Lie algebra. For a reductive Lie algebra $\mathfrak{g}$ the canonical homomorphism $\mathscr{T}_\Lambda(\mathfrak{g}) \to H^*(\mathfrak{g})$ is an isomorphism so the cohomology of $\mathfrak{g}$ may be identified with the algebra $\mathscr{T}_\Lambda(\mathfrak{g})$ of invariant forms. Furthermore if $\mathfrak{g}$ is reductive there is a finite dimensional graded subspace $P = P(\mathfrak{g})$ of $\mathscr{T}_\Lambda(\mathfrak{g})$ the dimension of which is the rank of $\mathfrak{g}$, which has only homogeneous elements of odd degree ($P = \oplus_{k \geq 0} P^{2k+1}$) and which is such that the graded commutative connected algebra $\mathscr{T}_\Lambda(\mathfrak{g})$ is *freely* generated by any homogeneous basis of $P(\mathfrak{g})$, [16]. Thus in this case $\mathscr{T}_\Lambda(\mathfrak{g})$ is isomorphic with the exterior algebra $\Lambda P(\mathfrak{g})$. The elements of $P(\mathfrak{g})$ are called *primitive forms on* $\mathfrak{g}$.

d) *The infinite dimensional case.* We will be interested in the Lie algebra of the group of gauge transformations which is infinite dimensional, so let us say a few words on the infinite dimensional case. If $\mathfrak{g}$ is infinite dimensional, it is again true that the Lie bracket defines a linear mapping from $\Lambda^2 \mathfrak{g}$ in $\mathfrak{g}$ and that, by duality we have a linear mapping $\mathrm{d} : \mathfrak{g}^* \to (\Lambda^2 \mathfrak{g})^*$ from the linear forms in the antisymmetric bilinear forms on $\mathfrak{g}$ by writing $\mathrm{d}\omega(X, Y) = -\omega([X, Y])$, $\forall \omega \in \mathfrak{g}^*$ and $\forall X$, $Y \in \mathfrak{g}$, but now $\Lambda^2 \mathfrak{g}^*$ is only a subspace of $(\Lambda^2 \mathfrak{g})^*$ and $\mathrm{d}\omega$ does not belong generally to this subspace. So one introduces the spaces $C^n(\mathfrak{g})$ of n-linear antisymmetric forms on $\mathfrak{g}$ and one defines on the direct sum $C^*(\mathfrak{g}) = \oplus_{n \in \mathbb{N}} C^n(\mathfrak{g})$ a structure of graded-commutative differential algebra by writing

$$(\alpha \cdot \beta)(X_1,\dots, X_{r+s}) = \frac{1}{(r+s)!} \sum_{\pi \in \mathfrak{S}_{r+s}} (-1)^{\epsilon(\pi)} \alpha(X_{\pi(1)},\dots, X_{\pi(r)})$$

$$\cdot \beta(X_{\pi(r+1)},\dots, X_{\pi(r+s)}),$$

for $\alpha \in C^r(\mathfrak{g})$, $\beta \in C^s(\mathfrak{g})$, $X_1, \dots, X_{r+s} \in \mathfrak{g}$, $\mathfrak{S}_{r+s}$ being the group of permutations of $r+s$ objects and $\epsilon(\pi)$ being the signature of $\pi \in \mathfrak{S}_{r+s}$, and by writing for $\alpha \in C^r(\mathfrak{g})$

$$\mathrm{d}\alpha(X_1,\dots, X_{r+1}) = \sum_{1 \leqslant p < q \leqslant r+1} (-1)^{p+q} \alpha([X_p, X_q], \hat{X}_1,\dots,\hat{X}_q,\dots,X_{r+1}).$$

Again $\mathrm{d}^2 = 0$ is just the Jacobi identity. $\Lambda\mathfrak{g}^*$ is a graded subalgebra of $C^*(\mathfrak{g})$ which coincides with $C^*(\mathfrak{g})$ when $\mathfrak{g}$ is finite dimensional; but when, $\mathfrak{g}$ is infinite dimensional, $\Lambda\mathfrak{g}^*$ is a strict subalgebra of $C^*(\mathfrak{g})$ which is generally not stable by d. In any case the cohomology $H^*(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the cohomology of $C^*(\mathfrak{g})$.

As it is well known (see e.g. in [11]) these definitions may be generalized in order to include the cohomology of $\mathfrak{g}$ with values in representations. Namely given a linear representation θ of $\mathfrak{g}$ in a vector space $\mathcal{M}$, one defines the space $C^n(\mathfrak{g},\mathcal{M})$ of *n-cochains of $\mathfrak{g}$ with values in $\mathcal{M}$* to be the space of n-linear antisymmetric mapping of $\mathfrak{g}$ in $\mathcal{M}$, one introduces then on the graded vector space $C^*(\mathfrak{g},\mathcal{M}) = \oplus_{n \in \mathbb{N}} C^n(\mathfrak{g},\mathcal{M})$ a linear mapping d of degree one by

$$\mathrm{d}\alpha(X_0,\dots, X_n) = \sum_{o \leqslant k \leqslant n} (-1)^k \theta(X_k)\alpha(X_0,\dots, \hat{X}_k,\dots, X_n) +$$

$$+ \sum_{o \leqslant l < m \leqslant n} (-1)^{l+m} \alpha([X_l, X_m], X_o,\dots, \hat{X}_l,\dots, \hat{X}_m,\dots, X_n)$$

for $\alpha \in C^n(\mathfrak{g},\mathcal{M})$ and $X_0, \dots, X_n \in \mathfrak{g}$. One has again $\mathrm{d}^2 = 0$ so $H^*(\mathfrak{g},\mathcal{M}) = \mathrm{Ker}(\mathrm{d})/\mathrm{Im}(\mathrm{d})$ is a graded vector space which is called *the cohomology of $\mathfrak{g}$ with value in $\mathcal{M}$*. In the case where $\mathcal{M}$ is an algebra $\mathbf{A}$ and θ is a homomorphism of $\mathfrak{g}$ in the Lie algebra of derivations of $\mathbf{A}$ then $C^*(\mathfrak{g}, \mathbf{A})$ is in a natural way a graded differential algebra so $H^*(\mathfrak{g}, \mathbf{A})$ is a graded algebra; when $\mathbf{A}$ is commutative $C^*(\mathfrak{g}, \mathbf{A})$ and therefore $H^*(\mathfrak{g}, \mathbf{A})$ are graded-commutative.

1.3. H. Cartan's notion of operation of a Lie algebra

Let G be a finite dimensional connected Lie group and P be a G-principal bundle over a (finite dimensional) manifold M. The projection $\pi : P \to M$ induces an injective homomorphism of differential algebras, $\pi^* : \Omega(M) \to \Omega(P)$, from the algebra $\Omega(M)$ of differential forms on M into the algebra $\Omega(P)$ of differential forms on P. Thus $\Omega(M)$ naturally identifies with the differential subalgebra $\pi^*(\Omega(M))$ of $\Omega(P)$; this subalgebra is called *the algebra of basic forms on P.* A differential form on P is basic if and only if it is invariant by the (right) action of G on P and horizontal, (i.e. it vanishes whenever one of the vectors on P to which it is applied is vertical). Therefore $\pi^*(\Omega(M)) \cong \Omega(M)$ is algebraically specified in $\Omega(P)$. A convenient way to describe the situation is the following one. The infinitesimal action of G on P gives an injective homomorphism from the Lie algebra Lie (G) of G into the Lie algebra of vector fields on P. Furthermore, the image of Lie (G) spans at each point of P the tangent space to the fiber (i.e. the space of vertical vectors). For any $X \in$ Lie (G) let i_X and L_X denote respectively the interior antiderivative and the Lie derivative on $\Omega(P)$ by the corresponding vector field. We have the usual relations

$$L_X = \mathrm{d}i_X + i_X\mathrm{d} \tag{1}$$

$$L_{[X,Y]} = L_X L_Y - L_Y L_X \tag{2}$$

$$i_{[X,Y]} = L_X i_Y - i_Y L_X. \tag{3}$$

An element $\omega \in \Omega(P)$ is invariant by the action of G on P iff. $L_X\omega = 0$ $\forall X \in$ $\in$ Lie (G), it is horizontal iff. $i_X\omega = 0$ $\forall X \in$ Lie (G), so, ω is a basic form iff. $L_X\omega = 0$ and $i_X\omega = 0$ for any $X \in$ Lie (G).

Following H. Cartan [10], [11], let us generalize the above structure relating Lie (G) and $\Omega(P)$ by the following definition.

DEFINITION 1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\mathscr{A}$ be a graded commutative differential algebra. One says that $\mathfrak{g}$ *operates on* $\mathscr{A}$ if we have a linear mapping i of $\mathfrak{g}$ in the antiderivations of degree -1 of $\mathscr{A}$ such that if for $X \in \mathfrak{g}$ we define the derivation L_X of degree 0 of $\mathscr{A}$ by

$$L_X = \mathrm{d}i_X + i_X\mathrm{d}, \tag{1}$$

we have for any $X, Y \in \mathfrak{g}$

$$L_{[X,Y]} = L_X L_Y - L_Y L_X \tag{2}$$

$$i_{[X,Y]} = L_X i_Y - i_Y L_X. \tag{3}$$

The pair $(\mathscr{A}, i)$, or simply $\mathscr{A}$ when there is no confusion, will be called a $\mathfrak{g}$-*-operation*. If $(\mathscr{A}, i)$ and $(\mathscr{A}', i')$ are two $\mathfrak{g}$-operations, a homomorphism ψ : $: \mathscr{A} \to \mathscr{A}'$ of differential algebras will be called a *homomorphism of* $\mathfrak{g}$-operations if we have:

$$i'_X \psi(\omega) = \psi(i_X \omega), \quad \forall X \in \mathfrak{g} \quad \text{and} \quad \forall \omega \in \mathscr{A}. \qquad \blacksquare$$

As already seen if G is a connected Lie group and P is a G-principal bundle then $\Omega(P)$ is canonically a Lie (G)-operation. If P' is another G-principal bundle then an homomorphism of Lie (G)-operations, $\psi : \Omega(P) \to \Omega(P')$, just corresponds to a G-principal bundle homomorphism, $\alpha : P' \to P$, by pull-back, i.e. $\psi = \alpha^*$. Therefore the above notion of operation generalizes the notion of principal bundle. We now want to generalize correspondingly the notion of connection. To do that one notices that a connection on a G-principal bundle P is given by a *connection form*; this is a Lie (G)-valued differential 1-form A, i.e. $A \in \text{Lie}\,(G) \otimes \Omega^1(P)$, such that $i_X A = X$ (verticality) and $L_X A = AX - XA$ (equivariance) for any $X \in \text{Lie}\,(G)$. This generalizes immediately; let $(\mathscr{A}, i)$ be $\mathfrak{g}$-operation, *an algebraic connection on* $\mathscr{A}$ (or simply a connection) will be an element A of $\mathfrak{g} \otimes \mathscr{A}^1$ such that $i_X A = X$ and $L_x A = [A, X]$ for any $X \in \mathfrak{g}$. In the above formula i_X, L_X are defined on $\mathfrak{g} \otimes \mathscr{A}$ by $i_X(Y \otimes \alpha) = Y \otimes i_X \alpha$ and $L_X(Y \otimes \alpha) = Y \otimes L_X \alpha$ for X, $Y \in \mathfrak{g}$ and $\alpha \in \mathscr{A}$ and one defines a bracket on $\mathfrak{g} \otimes \mathscr{A}$ extending the Lie bracket of $\mathfrak{g}$ by $[X \otimes \alpha,\ Y \otimes \beta] = [X, Y] \otimes \alpha \cdot \beta$ for X, $Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathscr{A}$; similarly one defines d on $\mathfrak{g} \otimes \mathscr{A}$ by $\mathrm{d}(X \otimes \alpha) = X \otimes \mathrm{d}\alpha$ for $X \in \mathfrak{g}$ and $\alpha \in \mathscr{A}$. With these notations one defines the curvature F of A to be the element $F = \mathrm{d}A + \frac{1}{2}\ [A, A]$ of $\mathfrak{g} \otimes \mathscr{A}^2$. From the definitions it follows that we have $i_X F = 0$ (horizontality) and $L_X F = [F, X]$ equivariance for $X \in \mathfrak{g}$.

Let $\mathscr{A}$ be an arbitrary $\mathfrak{g}$-operation and define the following subspaces of

$$\mathscr{H}(\mathscr{A}) = \{\alpha \in \mathscr{A} \mid i_X \alpha = 0, \quad \forall X \in \mathfrak{g}\}$$

$$\mathscr{T}(\mathscr{A}) = \{\alpha \in \mathscr{A} \mid L_X \alpha = 0, \quad \forall X \in \mathfrak{g}\}$$

$$\mathscr{B}(\mathscr{A}) = \{\alpha \in \mathscr{A} \mid i_X \alpha = 0, \ \text{and} \quad L_X \alpha = 0, \quad \forall X \in \mathfrak{g}\} = \\ = \mathscr{H}(\mathscr{A}) \cap \mathscr{T}(\mathscr{A}).$$

$\mathscr{H}(\mathscr{A})$ is a graded subalgebra of $\mathscr{A}$ which is stable by L_X, for $X \in \mathfrak{g}$, $\mathscr{T}(\mathscr{A})$ is a graded differential subalgebra of $\mathscr{A}$, $\mathscr{B}(\mathscr{A})$ is a graded differential subalgebra of $\mathscr{T}(\mathscr{A})$ and therefore also of $\mathscr{A}$. The elements of $\mathscr{H}(\mathscr{A})$ are called *horizontal elements of* $\mathscr{A}$, the elements of $\mathscr{T}(\mathscr{A})$ are called *invariant elements of* $\mathscr{A}$ and the elements of $\mathscr{B}(\mathscr{A})$ are called *basic elements of* $\mathscr{A}$. This terminology of course comes from the example where $\mathfrak{g} = \text{Lie}\,(G)$ with G connected and $\mathscr{A} = \Omega(P)$ where P is a G-principal bundle.

In the following, we shall have to consider operations which are not of the type $\Omega(P)$, (P a principal bundle). Notice that, in 1.2 c), we already met a $\mathfrak{g}$-operation which is not of the type $\Omega(P)$, namely $\Lambda\mathfrak{g}^*$, and that, furthermore, there is on this $\mathfrak{g}$-operation $\Lambda\mathfrak{g}^*$ a canonical connection, namely the identity mapping of $\mathfrak{g}$ on itself considered as an element of $\mathfrak{g}\otimes\mathfrak{g}^* = \mathfrak{g}\otimes\Lambda^1\mathfrak{g}^* \subset \mathfrak{g}\otimes\Lambda\,\mathfrak{g}^*$.

Let $\mathscr{A}$ be a $\mathfrak{g}$-operation, let E be a set and let Map $(E,\mathscr{A})$ be the set of mapping $f: E\to \mathscr{A}$ of E in $\mathscr{A}$; Map $(E,\mathscr{A})$ is also a $\mathfrak{g}$-operation if we define the graduation by $\mathrm{Map}^n(E,\mathscr{A}) \Leftrightarrow \mathrm{Map}(E,\mathscr{A}^n)$, the product by $(f\cdot g)(e) = f(e)\cdot g(e)$ for $e\in E$ and $f, g\in \mathrm{Map}(E,\mathscr{A})$ (i.e. the pointwise product), the differential by $(\mathrm{d}f)(e) = \mathrm{d}(f(e))$ for $e\in E$ and $f\in\mathrm{Map}(E,\mathscr{A})$ and if we define i_X by $(i_X f)(e) = i_X(f(e))$ for $X\in\mathfrak{g}$, $e\in E$ and $f\in\mathrm{Map}(E,\mathscr{A})$. Let us assume that $\mathscr{A}$ admits connections, (i.e. there is at least one). The set $\mathcal{C}$ of all connections on $\mathscr{A}$ is canonically an affine space; let us consider the $\mathfrak{g}$-operation $\mathrm{Map}(\mathcal{C},\mathscr{A})$. We claim that there is *a canonical connection* A on Map $(\mathcal{C},\mathscr{A})$ which is defined by the following; A is the identity mapping of $\mathcal{C}$ on itself considered as a mapping of $\mathcal{C}$ in $\mathfrak{g}\otimes\mathscr{A}^1$ by using (or by composition with) the inclusion $\mathcal{C}\subset\mathfrak{g}\otimes\mathscr{A}^1$. Thus A so defined belongs to $\mathrm{Map}(\mathcal{C},\mathfrak{g}\otimes\mathscr{A}^1) = \mathfrak{g}\otimes\mathrm{Map}(\mathcal{C},\mathscr{A}^1) = \mathfrak{g}\otimes\mathrm{Map}^1(\mathcal{C},\mathscr{A})$ and one easily verifies that it is a connection on Map $(\mathcal{C},\mathscr{A})$, (i.e. $i_X A = X$ and $L_X A = [A, X]$, $\forall X\in\mathfrak{g}$). Since $\mathcal{C}$ is an affine space, there is a well defined notion of *polynomial mapping* of $\mathcal{C}$ in $\mathscr{A}$; let us denote by $\mathcal{P}(\mathcal{C},\mathscr{A})$ the space of all these polynomial mappings. $\mathcal{P}(\mathcal{C},\mathscr{A})$ is a graded differential subalgebra of Map $(\mathcal{C},\mathscr{A})$ and the operation of $\mathfrak{g}$ on Map $(\mathcal{C},\mathscr{A})$ restricts to $\mathcal{P}(\mathcal{C},\mathscr{A})$ so $\mathcal{P}(\mathcal{C},\mathscr{A})$ is a $\mathfrak{g}$-operation; furthermore $A\in\mathfrak{g}\otimes\mathcal{P}(\mathcal{C},\mathscr{A})$ so it is a connection on $\mathcal{P}(\mathcal{C},\mathscr{A})$, (again called the canonical connection of $\mathcal{P}(\mathcal{C},\mathscr{A})$). There is yet a smaller $\mathfrak{g}$-operation with connection contained in Map $(\mathcal{C},\mathscr{A})$ which we now describe. Let A^α be the components of the canonical connection with respect to some basis E_α of $\mathfrak{g}$, (i.e. $A = E_\alpha\otimes A^\alpha$), $A^\alpha\in\mathrm{Map}(\mathcal{C},\mathscr{A}^1) = \mathrm{Map}^1(\mathcal{C},\mathscr{A})$, and let $\mathcal{B}^{*,0}(\mathscr{A})$ be the smallest differential subalgebra of Map $(\mathcal{C},\mathscr{A})$ containing the A^α's. $\mathcal{B}^{*,0}(\mathscr{A})$ is generated by the A^α's and the $\mathrm{d}A^\alpha$'s which belong to $\mathcal{P}(\mathcal{C},\mathscr{A})$ so $\mathcal{B}^{*,0}(\mathscr{A})\subset\mathcal{P}(\mathcal{C},\mathscr{A})$. Moreover, $\mathcal{B}^{*,0}(\mathscr{A})$ is stable by the operation of $\mathfrak{g}$ and, by construction A is in $\mathfrak{g}\otimes\mathcal{B}^{1,0}(\mathscr{A})$ so, since $\mathcal{B}^{*,0}(\mathscr{A}) = \bigoplus_{n\in\mathbb{N}}\mathcal{B}^{n,0}(\mathscr{A})$ is a graded subalgebra of $\mathcal{P}(\mathcal{C},\mathscr{A})$, $\mathcal{B}^{*,0}(\mathscr{A})$ is a $\mathfrak{g}$-operation with connection A.

All these constructions apply when $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathscr{A} = \Omega(P)$ where G is a connected Lie group and P is a G-principal bundle over M. In this case, we denote $\mathcal{B}^{*,0}(\Omega(P))$ simply by $\mathcal{B}^{*,0}(P)$ and there is another natural operation $\tilde{\mathcal{B}}^{*,0}(P)$ which is a little bigger than $\mathcal{B}^{*,0}(P)\cdot\tilde{\mathcal{B}}^{*,0}(P)$ is the set of all polynomial mapping α of $\mathcal{C}$ in $\Omega(P)$ such that the value at $\zeta\in P$ of $\alpha(a)$ for $a\in\mathcal{C}$ only depends on the values at ζ of a and of a finite number of derivatives of a ($a\in\mathrm{Lie}(G)\otimes\Omega^1(P)$). In the case where P is the trivial bundle $G\times M$ with $\dim(M) = n$, integration of $\in\tilde{\mathcal{B}}^{n,0}(G\times M)$ on M yields «local polynomial functionals» on $\mathcal{C}$ via (here M is identified to $e\times M$, e is the unit of G)

$$F(a) = \int_M \alpha(a),\quad \alpha\in\tilde{\mathcal{B}}^{n,0}(G\times M),\quad a\in\mathcal{C}.$$

1.4. The Weil algebra of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\psi : \mathscr{A} \to \mathscr{A}'$ be a homorphism of $\mathfrak{g}$-operations; we again denote by ψ the linear mapping of $\mathfrak{g} \otimes \mathscr{A}$ in $\mathfrak{g} \otimes \mathscr{A}'$ defined by $\psi(X \otimes \omega) = X \otimes \psi(\omega)$ for $X \in \mathfrak{g}$ and $\omega \in \mathscr{A}$. With these notations, if A is a connection on $\mathscr{A}$ then $\psi(A)$ is obviously a connection on $\mathscr{A}'$ which will be called *the image of A by ψ*. If $\mathscr{A}$ and $\mathscr{A}'$ are equipped with connections A and A' respectively the homomorphism of $\mathfrak{g}$-operations $\psi : \mathscr{A} \to \mathscr{A}'$ will be called a *homomorphism of $\mathfrak{g}$-operations with connections* if $A' = \psi(A)$. It turns out that, in the category of $\mathfrak{g}$-operations with connections, there is a universal initial object, called *the Weil algebra of* $\mathfrak{g}$ and denoted by $W(\mathfrak{g})$, which we now describe.

As usual here we consider the symmetric algebra $S\mathfrak{g}^*$ over the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, (i.e. the algebra of polynomials on $\mathfrak{g}$), to be evenly graded by writing $(S\mathfrak{g}^*)^{2n} = S^n\mathfrak{g}^*$ and $(S\mathfrak{g}^*)^{2n+1} = \{0\}$. With this convention $S\mathfrak{g}^*$ is a graded commutative algebra and we define the graded commutative algebra $W(\mathfrak{g})$ by $W(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$. In the following (E_α) will be an arbitrary but fixed base of $\mathfrak{g}$ with dual basis (E^α). Introducing the elements A^α and F^α of $W(\mathfrak{g})$ defined by $A^\alpha = E^\alpha \otimes \mathbb{1}$ and $F^\alpha = \mathbb{1} \otimes E^\alpha$, we see that $W(\mathfrak{g})$ is just the free connected graded commutative algebra generated by A^α's in degree one and the F^α's in degree two. Let us introduce the elements A and F of $\mathfrak{g} \otimes W(\mathfrak{g})$ by $A = \sum_\alpha E_\alpha \otimes A^\alpha$ and $F = \sum_\alpha E_\alpha \otimes F^\alpha$ and define $\mathrm{d}A^\alpha$ and $\mathrm{d}F^\alpha$ by $\mathrm{d}A = \sum_\alpha E_\alpha \otimes \mathrm{d}A^\alpha$ and $\mathrm{d}F = \sum_\alpha E_\alpha \otimes \mathrm{d}F^\alpha$ with $\mathrm{d}A = -\frac{1}{2}[A, A] + F$ and $\mathrm{d}F = -[A, F]$. d extends uniquely as antiderivation of $W(\mathfrak{g})$; d is of degree one and $\mathrm{d}^2 = 0$, (since $\mathrm{d}^2A^\alpha = 0$ and $\mathrm{d}^2F^\alpha = 0$), so $W(\mathfrak{g})$ equipped with this differential is a (free) graded commutative differential algebra. One defines, for $X \in \mathfrak{g}$, and antiderivation i_X of $W(\mathfrak{g})$ by $i_X(a^\alpha) = X^\alpha$ and $i_X(F^\alpha) = 0$. It is straightforward to verify that, equipped with i, $W(\mathfrak{g})$ is a $\mathfrak{g}$-operation and that A is a connection on it with curvature equal to F; this $\mathfrak{g}$-operation with connection is called *the Weil algebra of* $\mathfrak{g}$. One has the following theorem.

THEOREM 2. (Universal property of $W(\mathfrak{g})$). *For any $\mathfrak{g}$-operation with connection $\mathscr{A}$, there is a unique homomorphism of $\mathfrak{g}$-operations with connections of $W(\mathfrak{g})$ in $\mathscr{A}$.*

Indeed if $a = \sum_\alpha E_\alpha \otimes a^\alpha$ denotes the connection of $\mathscr{A}$ and $f = \sum_\alpha E_\alpha \otimes f^\alpha$ is its curvature, any homomorphism ψ of $\mathfrak{g}$-operation with connections of $W(\mathfrak{g})$ in $\mathscr{A}$ must satisfy $\psi(A^\alpha) = a^\alpha$ (the connection is mapped on the connection) an $\psi(F^\alpha) = f^\alpha$ (the differential of A^α is mapped on the differential of a^α). Now, by the universal properties of the exterior algebra, the symmetric algebra and the tensor product, there is a unique homomorphism of algebra of $W(\mathfrak{g})$ in $\mathscr{A}$ satisfying the above conditions; this homomorphism ψ is, as easily seen, a homomorphism of $\mathfrak{g}$-operations with connections. This unique $\psi : W(\mathfrak{g}) \to \mathscr{A}$ will be called *the canonical homomorphism of* $W(\mathfrak{g})$ *in* $\mathscr{A}$. ■

Although the differential of $W(\mathfrak{g})$ restricted to $\Lambda\mathfrak{g}^* \otimes \mathbb{1}$ does not coincide with the one defined in 1.2 a) on $\Lambda\mathfrak{g}^*$ (since $\Lambda\mathfrak{g}^* \otimes \mathbb{1}$ is not stable), the derivations $L_X = i_X \mathrm{d} + \mathrm{d} i_X$ $(X \in \mathfrak{g})$ coincide on $\Lambda\mathfrak{g}^* \otimes \mathbb{1}$ with the L_X's defined in 1.2 c) on $\Lambda\mathfrak{g}^*$; i.e. it is induced by the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Similarily one verifies that $\mathbb{1} \otimes S\mathfrak{g}^*$ is stable by the L_X and that the corresponding derivations of $S\mathfrak{g}^*$ are also induced by the coadjoint action $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $\mathscr{T}_W(\mathfrak{g})$ be the set of invariant elements of $W(\mathfrak{g})$ (i.e. $\mathscr{T}_W(\mathfrak{g}) = \mathscr{T}(W(\mathfrak{g}))$ and let $\mathscr{T}_S(\mathfrak{g})$ be the set of invariant polynomials on $\mathfrak{g}$. We have $\mathscr{T}_\Lambda(\mathfrak{g}) \otimes \mathscr{T}_S(\mathfrak{g}) \subset \mathscr{T}_W(\mathfrak{g})$, (in fact $\mathscr{T}_\Lambda(\mathfrak{g}) \otimes \mathbb{1} = (\Lambda\mathfrak{g}^* \otimes \mathbb{1}) \cap \mathscr{T}_W(\mathfrak{g})$, $\mathbb{1} \otimes \mathscr{T}_S(\mathfrak{g}) = (\mathbb{1} \otimes S\mathfrak{g}^*) \cap \mathscr{T}_W(\mathfrak{g})$ as shown above), but is is worth noticing that *this inclusion is strict*.

From the very definitions, it follows that $\mathbb{1} \otimes S\mathfrak{g}^*$ is the set of all horizontal elements of $W(\mathfrak{g})$, (i.e. $\mathbb{1} \otimes S\mathfrak{g}^* = \mathscr{H}(W(\mathfrak{g}))$), so the set of all basic elements of $W(\mathfrak{g})$ is just $\mathbb{1} \otimes \mathscr{T}_S(\mathfrak{g})$, (i.e. $\mathbb{1} \otimes \mathscr{T}_S(\mathfrak{g}) = \mathscr{B}(W(\mathfrak{g}))$). Furthermore $\mathbb{1} \otimes \mathscr{T}_S(\mathfrak{g})$ consists of cocycles of $W(\mathfrak{g})$ and we have the following result.

THEOREM 3. *The following conditions are equivalent for* $\omega \in W(\mathfrak{g})$

(i) ω *is basic*

(ii) ω *is of the form* $\mathbb{1} \otimes P$ *with* $P \in \mathscr{T}_S(\mathfrak{g})$

(iii) ω *is of the form* $\mathbb{1} \otimes P$ *and* $\mathrm{d}\omega = 0$.

Let $\mathscr{A}$ be a $\mathfrak{g}$-operation and let us denote by $H_B(\mathscr{A})$, and call *basic cohomology of* $\mathscr{A}$, the cohomology algebra of the graded commutative differential algebra $\mathscr{B}(\mathscr{A})$ of basic elements of $\mathscr{A}$. From theorem 3, it follows that we have $H_B(W(\mathfrak{g})) = \mathscr{B}(W(\mathfrak{g})) = \mathscr{T}_S(\mathfrak{g})$ with $H_B^{2k}(W(\mathfrak{g})) = \mathscr{T}_S^k(\mathfrak{g})$ and $H_B^{2k+1}(W(\mathfrak{g})) = \{0\}$, where $\mathscr{T}_S^k(\mathfrak{g})$ denotes the space of homogeneous invariant polynomials of degree k on $\mathfrak{g}$, (remembering that the corresponding elements of $\mathscr{B}(W(\mathfrak{g})) \subset W(\mathfrak{g})$ are of degree $2k$ for the graduation of $W(\mathfrak{g})$). One has the following theorem.

THEOREM 4. (Weil homomorphism). *Let* a_0 *and* a_1 *be two connections on the* $\mathfrak{g}$*-operation* $\mathscr{A}$ *and let* ψ_0 *and* ψ_1 *be the corresponding canonical homomorphisms of* $W(\mathfrak{g})$ *in* $\mathscr{A}$ *with* $\psi_0(A) = a_0$, $\psi_1(A) = a_1$. *Then the corresponding induced homomorphisms* ψ_0^b *and* ψ_1^b *in basic cohomology coincide.*

I.e. one has, whenever $\mathscr{A}$ admits connections, a homomorphism $w : \mathscr{T}_S(\mathfrak{g}) \to$ $\to H_B(\mathscr{A})$, with $w(\mathscr{T}_S^k(\mathfrak{g})) \subset H_B^{2k}(\mathscr{A})$, which is called *the Weil homomorphism* and is induced by the canonical homomorphism of $W(\mathfrak{g})$ in $\mathscr{A}$ associated with any connection on $\mathscr{A}$.

The proof of this theorem goes as follows. One introduces the one-parameter family of connections $a_t = (1-t)a_0 + t a_1$, $t \in [0,1]$, with curvature $\mathfrak{f}_t$ and the corresponding $\psi_t : W(\mathfrak{g}) \to \mathscr{A}$ with $\psi_t(A) = a_t$; by restriction to $\mathbb{1} \otimes \mathscr{T}_S(\mathfrak{g})$ one has homomorphisms $\tilde{\psi}_t : \mathscr{T}_S^k(\mathfrak{g}) \to \mathscr{B}^{2k}(\mathscr{A})$, $\tilde{\psi}_t(\mathscr{T}_S^k(\mathfrak{g}))$ consisting of cocycles. For $P \in \mathscr{T}_S^k(\mathfrak{g})$, $\tilde{\psi}_t(P) = P(\mathfrak{f}_t, \dots, \mathfrak{f}_t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\psi}_t(P) = \mathrm{d}(kP(a_1 - a_0, \mathfrak{f}_t, \dots, \mathfrak{f}_t))$, where $P(a_1 - a_0, \mathfrak{f}_t, \dots, \mathfrak{f}_t)$ is, as easily verified, in $\mathscr{B}(\mathscr{A})$. therefore one has $\tilde{\psi}_1(P) - \tilde{\psi}_0(P) = \mathrm{d}\int_0^1 k\,P(a_1 - a_0, \mathfrak{f}_t, \dots, \mathfrak{f}_t)\,\mathrm{d}t$ which shows that $\tilde{\psi}_1(P)$ and $\tilde{\psi}_0(P)$ have the same image in $H_B(\mathscr{A})$. ∎

It is well known that a G-principal bundle P admits connections, or which is the same, the Lie (G)-operation $\Omega(P)$ admits connections; in this case the above result is the familiar Weil homomorphism (see in [17] for instance).

One has $W(\mathfrak{g}) = (\Lambda\mathfrak{g}^* \otimes \mathbb{1}) \oplus (\Lambda\mathfrak{g}^* \otimes S^+\mathfrak{g}^*)$, where $S^+\mathfrak{g}^* = \underset{k \geqslant 1}{\oplus} S^k\mathfrak{g}^*$; furthermore $\Lambda\mathfrak{g}^* \otimes S^+\mathfrak{g}^*$ is a graded ideal in $W(\mathfrak{g})$ stable by the differential and by the i_X's $(X \in \mathfrak{g})$. $W(\mathfrak{g})/\Lambda\mathfrak{g}^* \otimes S^+\mathfrak{g}^*$ is canonically isomorphic as graded algebra to $\Lambda\mathfrak{g}^*$ and it is easy to see that the corresponding canonical projection $p : W(\mathfrak{g}) \to \Lambda\mathfrak{g}^*$ is an homomorphism of graded differential algebras for the structure of differential algebra defined on $\Lambda\mathfrak{g}^*$ in 1.2 a), and that p is in fact a homomorphism of $\mathfrak{g}$-operations, for the structure defined on $\Lambda\mathfrak{g}^*$ in 1.2 c) which maps the connection of $W(\mathfrak{g})$ on the canonical (flat) connection of $\Lambda\mathfrak{g}^*$ defined in 1.3. We shall refer to this surjective homomorphism of $\mathfrak{g}$-operations (with connections), $p : W(\mathfrak{g}) \to \Lambda\mathfrak{g}^*$ as the *canonical projection of* $W(\mathfrak{g})$ *on* $\Lambda\mathfrak{g}^*$.

Notice that an alternative system of homogeneous free system of generators for $W(\mathfrak{g})$ is the A^α's and the $\mathrm{d}A^\alpha$'s; therefore $W(\mathfrak{g})$ identifies as graded commutative differential algebra with the contractible algebra $\underset{\alpha}{\otimes}\mathscr{C}(A^\alpha, \mathrm{d}A^\alpha)$, (see in 1.1), so its cohomology is trivial. One has more generally the following theorem.

THEOREM 5. *The cohomology algebras $H(W(\mathfrak{g}))$ and $H(\mathscr{T}_W(\mathfrak{g}))$ of $W(\mathfrak{g})$ and of the subalgebra $\mathscr{T}_W(\mathfrak{g})$ are trivial.*

I.e. one has $H^k(W(\mathfrak{g})) = 0$ and $H^k(\mathscr{T}_W(\mathfrak{g})) = 0$ for $k \geqslant 1$ and, of course $H^0(W(\mathfrak{g}))$ and $H^0(\mathscr{T}_W(\mathfrak{g}))$ identify with the ground field $\mathbb{K}$. ∎

We already known the result for $H(W(\mathfrak{g}))$ and to show it for $H(\mathscr{T}_W(\mathfrak{g}))$

consider the unique antiderivation h of $W(\mathfrak{g})$ satisfying $h(A^\alpha) = 0$ and $h(\mathrm{d}A^\alpha) = A^\alpha$. We have $hL_X = L_X h$ for $X \in \mathfrak{g}$ (since this holds on the generators A^α, $\mathrm{d}A^\alpha$) and the derivation $\mathrm{d}h + h\mathrm{d}$ satisfies $(\mathrm{d}h + h\mathrm{d})A^\alpha = A^\alpha$, $(\mathrm{d}h + h\mathrm{d})\mathrm{d}A^\alpha = \mathrm{d}A^\alpha$; so $\mathrm{d}h + h\mathrm{d}$ is the «degree in generators A^α, $\mathrm{d}A^\alpha$» and therefore h gives a contracting homotopy for d on $W(\mathfrak{g})^+$ and also on $\mathscr{T}_W(\mathfrak{g})^+$ since it commutes with the L_X's ($X \in \mathfrak{g}$). ■

Let P be an invariant polynomial on $\mathfrak{g}$, $P \in \mathscr{T}_S(\mathfrak{g})$. Then $\mathbb{1} \otimes P \in W(\mathfrak{g})$ is closed and belongs to $\mathscr{T}_W(\mathfrak{g})$ in view of theorem 3 and therefore, is of the form $\mathbb{1} \otimes P = \mathrm{d}Q$ with $Q \in \mathscr{T}_W(\mathfrak{g})$ in view of theorem 5. Let us consider the image $p(Q)$ of Q by the canonical projection; $p(Q)$ is an invariant form on $\mathfrak{g}$, i.e $p(Q) \in \mathscr{T}_\Lambda(\mathfrak{g})$, since $Q \in \mathscr{T}_W(\mathfrak{g})$. If $Q' \in \mathscr{T}_W(\mathfrak{g})$ is such that $\mathbb{1} \otimes P = \mathrm{d}Q'$, then $\mathrm{d}(Q - Q') = 0$ so, again by theorem 5, $Q - Q' = \mathrm{d}L$ for some $L \in \mathscr{T}_W(\mathfrak{g})$. It follows that $p(Q) - p(Q') = \mathrm{d}p(L) = 0$, since $p(L)$ is an invariant form (see in 1.2 c)), and thus $p(Q)$ does only depend on P; one denotes by $\rho(P)$ this element $p(Q)$ of $\mathscr{T}_\Lambda(\mathfrak{g})$. This linear mapping $\rho : \mathscr{T}_S(\mathfrak{g}) \to \mathscr{T}_\Lambda(\mathfrak{g})$ from the invariant polynomials on $\mathfrak{g}$ in the invariant exterior forms on $\mathfrak{g}$ is *the Cartan map*. One has $\rho(\mathscr{T}^k_S(\mathfrak{g})) \subset \mathscr{T}^{2k-1}_\Lambda(\mathfrak{g})$ since $P \in \mathscr{T}^k_S(\mathfrak{g})$ implies $\mathbb{1} \otimes P \in \mathscr{T}^{2k}_W(\mathfrak{g})$ so Q as above is in $\mathscr{T}^{2k-1}_W(\mathfrak{g})$ and thus $\rho(P) = p(Q) \in \mathscr{T}^{2k-1}_\Lambda(\mathfrak{g})$. One has the following deep result [10], [18].

THEOREM 6. *Suppose that $\mathfrak{g}$ is a reductive Lie algebra. Then the image of the Cartan map, $\rho : \mathscr{T}_S(\mathfrak{g}) \to \mathscr{T}_\Lambda(\mathfrak{g})$, is the subspace $P(\mathfrak{g})$ of $\mathscr{T}_\Lambda(\mathfrak{g})$ of primitive forms on $\mathfrak{g}$ (see in 1.2 c)). The kernel of ρ is the space of decomposable elements of $\mathscr{T}_S(\mathfrak{g})$.*

I.e. elements which are linear combinations of products of several elements of $\mathscr{T}^+_S(\mathfrak{g})$. ■

DEFINITION 2. Let $\mathfrak{g}$ be a reductive Lie algebra. A *transgression* is a linear mapping $\tau : P(\mathfrak{g}) \to \mathscr{T}_S(\mathfrak{g})$ such that $\tau(P^{2k+1}(\mathfrak{g})) \subset \mathscr{T}^{k+1}_S(\mathfrak{g})$ and that $\rho \cdot \tau$ is the identity mapping of $P(\mathfrak{g})$ on itself (ρ being the Cartan map). ■

By choosing a homogeneous basis of $P(\mathfrak{g})$, one easily constructs such a transgression by fixing, for each element of the basis, an homogeneous invariant polynomial of the appropriate degree which is mapped by ρ on this element of the basis. Thus transgressions exist and are generally far to be unique. Let τ be a transgression and (ω_i) ($i \in \{1, 2, \ldots, \text{rank of}\}$) be a basis of $P = P(\mathfrak{g})$; then it follows from the last theorem that the $\tau(\omega_i)$'s (and $\mathbb{1}$) freely generate the algebra $\mathscr{T}_S(\mathfrak{g})$ of invariant polynomials on $\mathfrak{g}$, i.e. $\mathscr{T}_S(\mathfrak{g})$ identifies with the symmetric algebra $S\tau(P)$ over $\tau(P)$: $\mathscr{T}_S(\mathfrak{g}) = S\tau(P) \cong SP$.

Finally let us introduce the following terminology: An element Q of $\mathcal{T}_{W}(\mathfrak{g})$ is said to be a *transgression cochain* if dQ is in $\mathbf{1} \otimes \mathcal{T}_S(\mathfrak{g}) \subset W(\mathfrak{g})$.

A representation θ of $\mathfrak{g}$ in a vector space $\mathcal{M}$ is called *semi-simple* if any invariant subspace of $\mathcal{M}$(by $\theta(X)$, $X \in \mathfrak{g}$) admits a supplementary in $\mathcal{M}$ which is also invariant; the subspace of invariant elements of $\mathcal{M}$ will be denoted by $\mathcal{M}^I$. We saw that if $\mathfrak{g}$ is a reductive Lie algebra its cohomology identifies with $\mathcal{T}_\Lambda(\mathfrak{g})$; this generalizes to cohomology with values in $\mathcal{M}$, $H^*(\mathfrak{g}, \mathcal{M})$, for $\mathcal{M}$ semi-simple by the following [11]: If $\mathfrak{g}$ is reductive and $(\theta, \mathcal{M})$ semi-simple, $H^*(\mathfrak{g}, \mathcal{M})$ identifies with $\mathcal{M}^I \otimes \mathcal{T}_\Lambda(\mathfrak{g})$. This applies of course if $\mathcal{M}$ is the algebra $S\mathfrak{g}^*$ of polynomials on $\mathfrak{g}$, thus, for $\mathfrak{g}$ reductive $H^*(\mathfrak{g}, S\mathfrak{g}^*) = \mathcal{T}_S(\mathfrak{g}) \otimes \mathcal{T}_\Lambda(\mathfrak{g}) = S\tau(P) \otimes \Lambda P \cong SP \otimes \Lambda P$ (τ being a transgression).

Warning. As algebras $C^*(\mathfrak{g}, S\mathfrak{g}^*)$ and $W(\mathfrak{g})$ coincide ($\mathfrak{g}$ being finite dimensional) but their differential are different, $C^*(\mathfrak{g}, S\mathfrak{g}^*)$ is a minimal differential algebra (generated in degrees one and two) while $W(\mathfrak{g})$ is a contractible differential algebra. We will often use the symbol δ to denote the differential of $C^*(\mathfrak{g}, S\mathfrak{g}^*)$ in order to distinguish it from the differential of $W(\mathfrak{g})$ which we continue to denote by d. With the notations of the beginning of this paragraph δ reads $\delta A = -\frac{1}{2}[A, A]$ and $\delta F = -[A, F]$. Notice also that when we say that $C(\mathfrak{g}, S\mathfrak{g}^*)$ is a minimal algebra we use the graduation of the Weil algebra i.e. $C^m(\mathfrak{g}, S^n\mathfrak{g}^*) \subset W^{m+2n}(\mathfrak{g})$.

Let us consider the trivial G-principal bundle $M \times G$ where M is n-dimensional (n-dimensional space-time); then, by restriction to the section $x \mapsto (x, 1)$ corresponding to the trivialization, any element of $B^{*,0}(M \times G)$ gives an element of the subalgebra $B^{*,0} = \oplus_r B^{r,0}$ of the algebra $B^{*,*}$ described in the introduction. Furthermore, this mapping is a surjective homomorphism of graded algebra mapping the differential of $B^{*,0}(M \times G)$ on the restriction to $B^{*,0}$ of the differential d of $B^{*,*}$ and the connection of $B^{*,0}(M \times G)$ on the element $E_\alpha \otimes A^\alpha$ of Lie $(G) \otimes B^{*,0}$ (see the introduction). Thus one has, by composition, a homomorphism of W (Lie (G)) in $B^{*,0}$ mapping the connection of W (Lie (G)) on $E_\alpha \otimes A^\alpha$ which we call the canonical homomorphism. Notice that the L_X are naturally defined on $B^{*,0}$ and that the canonical homorphism intertwines the action of the L_X on W (Lie (G)) and on $B^{*,0}$. The canonical homomorphism of W (Lie (G)) in $B^{*,0}$ is of course the unique homomorphism of differential algebras mapping the connection of W (Lie (G)) on $E_\alpha \otimes A^\alpha$.

LEMMA 1. *The canonical homomorphism of* W (Lie (G)) *in* $B^{*,0}$ *is surjective and induces isomorphisms of vector spaces* W^r (Lie (G)) $\to B^{r,0}$ *for any* $r \leqslant n = \dim(M)$.

For the proof, let us notice that the surjectivity is obvious and that the $A_{\mathcal{W}}^{\alpha_1} \ldots A_{\mathcal{W}}^{\alpha_a}(F_{\mathcal{W}}^{\beta_1})^{m_1} \ldots (F_{\mathcal{W}}^{\beta_b})^{m_b}$ with $a + 2\sum_{i=1}^{i=b} m_i = r$, $\alpha_1 < \alpha_2 < \ldots < \alpha_a$, $\beta_1 < < \beta_2 < \ldots < \beta_b$ and α_k, $\beta_i \leqslant \dim(G)$ form a basis of $\mathcal{W}^r(\mathrm{Lie}(G))$, where $A_{\mathcal{W}}$ denotes the connection of $\mathcal{W}(\mathrm{Lie}(G))$ and $F_{\mathcal{W}}$ denotes its curvature, we shall show that the corresponding functionals of $\mathcal{C}$ in $\Omega^r(M)$ are linearily independent (which implies the lemma). For that, is sufficient to produce, for any α_k, β_l, m_i as above, an element α of $\mathcal{C}$ such that at some $x_0 \in M$, one has $\alpha^{\alpha_1}(x_0) \ldots \alpha^{\alpha_a}(x_0)$ $(\mathfrak{z}^{\beta_1}(x_0))^{m_1} \ldots (\mathfrak{z}^{\beta_b}(x_0))^{m_b} \neq 0$ and all the other products vanish at x_0. To construct such a $\alpha \in \mathcal{C}$, notice that given an arbitrary $\mathrm{Lie}(G)$-valued 1-form α_0 at x_0 and an arbitrary $\mathrm{Lie}(G)$-valued 2-form $\mathfrak{z}_0$ at x_0, there exists a $\mathrm{Lie}(G)$-valued 1-form α on M such that $\alpha(x_0) = \alpha_0$ and $\mathrm{d}\alpha(x_0) + \frac{1}{2}[\alpha(x_0), \alpha(x_0)] = \mathfrak{z}_0$. Thus, there is a $\alpha \in \mathcal{C}$ such that $\alpha^{\alpha_1}(x_0) = \mathrm{d}x^1, \ldots, \alpha^{\alpha_a}(x_0) = \mathrm{d}x^a$ and the other components of $\alpha(x_0)$ vanish,

$$\mathfrak{z}^{\beta_1}(x_0) = \frac{1}{m_1!} \sum_{k=a+1}^{k=a+m_1} \mathrm{d}x^k \wedge \mathrm{d}x^{k+m_1}, \ldots,$$

$$\mathfrak{z}^{\beta_b}(x_0) = \frac{1}{m_b!} \sum_{k=a+\sum_{i=1}^{b-1} 2m_i+1}^{k=a+\sum_{i=1}^{b-1} 2m_i+m_b} \mathrm{d}x^k \wedge \mathrm{d}x^{k+m_b}$$

and the other components of $\mathfrak{z}(x_0)$ vanish; (x^k) denotes a coordinates system around x_0. Clearly such a $\alpha \in \mathcal{C}$ satisfies the above conditions. ■

It follows immediately from lemma 1 that the canonical homomorphism of $\mathcal{W}(\mathrm{Lie}(G))$ in $\mathcal{B}^{*,0}(M \times G)$, which is surjective by construction, also induces isomorphisms $\mathcal{W}^r(\mathrm{Lie}(G)) \to \mathcal{B}^{r,0}(M \times G)$ for $r \leqslant \dim(M)$. Since all this comes from local considerations (in fact jets of finite orders) and since a principal G-bundle is locally trivializable one has the following theorem.

THEOREM 7. *Let P be a G-principal bundle over M. Then the canonical homomorphism of $\mathcal{W}(\mathrm{Lie}(G))$ in $\mathcal{B}^{*,0}(P)$ is surjective and induces isomorphism of vector spaces $\mathcal{W}^r(\mathrm{Lie}(G)) \to \mathcal{B}^{r,0}(P)$ for any $r \leqslant \dim(M)$.*

Notice that, apart surjectivity, nothing is said on what happens for $\dim(M) < < r \leqslant \dim(P) = \dim(M) + \dim(G)$.

2. B.R.S. ALGEBRAS AND THE WEIL-B.R.S. ALGEBRA OF A LIE ALGEBRA

2.1. Bigraded commutative differential algebras

A bigraded algebra will be an associative algebra $\mathscr{A}$ which admits the direct sum decomposition $\mathscr{A} = \underset{(r,s)\in \mathbb{N}^2}{\oplus} \mathscr{A}^{r,s}$ and is such that the product satisfies $\mathscr{A}^{r,s} \cdot \mathscr{A}^{r',s'} \subset \mathscr{A}^{r+r',s+s'}$. The elements of $\mathscr{A}^{r,s}$ are called *homogeneous for the bidegree* or *bihomogeneous* of *bidegree* (r, s). A linear mapping L of $\mathscr{A}$ in itself is said to be *bihomogeneous of bidegree* $(m, n) \in \mathbb{Z}^2$ if $L(\mathscr{A}^{r,s}) \subset \mathscr{A}^{r+m,s+n}$ for any $(r, s) \in \mathbb{N}^2$ (we extend the bigraduation to $\mathbb{Z}^2$ with the convention $\mathscr{A}^{r,s} = \{0\}$ if r or s is strictly negative). The elements of $\mathscr{A}^k = \underset{r+s=k}{\oplus} \mathscr{A}^{r,s}$ are called *homogeneous of total degree* k. Thus a bigraded algebra is, in particular, a graded algebra for the graduation corresponding to the total degree, so the various notions defined for graded algebras in 1.1 such as graded commutativity for instance, make sense for bigraded algebras. Notice also that $\mathscr{A}^{*,0} = \underset{r}{\oplus} \mathscr{A}^{r,0}$ and $\mathscr{A}^{0,*} = \underset{s}{\oplus} \mathscr{A}^{0,s}$ are graded subalgebras of $\mathscr{A}$. We shall denote by $P^{r,s} : \mathscr{A} \to \mathscr{A}^{r,s}$, the projection of $\mathscr{A}$ on $\mathscr{A}^{r,s}$ corresponding to the direct sum decomposition $\mathscr{A} = \underset{r,s}{\oplus} \mathscr{A}^{r,s}$.

Let $\mathscr{A}$ be a bigraded algebra and suppose that d is a differential on $\mathscr{A}$ considered as graded algebra. Then we have $\mathrm{d}\mathscr{A}^{r,s} \subset \underset{r'+s'=r+s+1}{\oplus} \mathscr{A}^{r'+s'}$; one introduces $\mathrm{d}^{1,0}$ and $\mathrm{d}^{0,1}$ by $\mathrm{d}^{1,0} = \sum_{r,s} P^{r+1,s} \circ \mathrm{d} \circ P^{r,s}$ and $\mathrm{d}^{0,1} = \sum_{r,s} P^{r,s+1} \circ \mathrm{d} \circ P^{r,s}$. In general one has $\mathrm{d} \neq \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$, (in fact $\mathrm{d} = \sum_{\{r,s,r',s' \text{ with } r'+s'=r+s+1\}} P^{r',s'} \circ \mathrm{d} \circ P^{r,s}$). In the case where $\mathrm{d} = \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$, $\mathscr{A}$ equipped with d is called a *bigraded differential algebra*; $\mathrm{d} = \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$ implies that $\mathrm{d}^{1,0}$ and $\mathrm{d}^{0,1}$ are two anticommuting differentials, $(\mathrm{d}^{1,0})^2 = 0$, $(\mathrm{d}^{0,1})^2 = 0$ and $\mathrm{d}^{1,0}\mathrm{d}^{0,1} + \mathrm{d}^{0,1}\mathrm{d}^{1,0} = 0$. Conversely if the bigraded algebra $\mathscr{A}$ is equipped with two anticommuting differentials $\mathrm{d}^{1,0}$ and $\mathrm{d}^{0,1}$ which are bihomogeneous of respective bidegrees (1,0) and (0,1), then $\mathscr{A}$ is a bigraded algebra with *total differential* $\mathrm{d} = \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$.

An example of bigraded commutative differential algebra is the algebra $\Omega(M)$ of complex differential forms on a complex manifold M; $\Omega^{r,s}(M)$ is then the space of differential forms which are of degree r in the $\mathrm{d}z^k$'s and of degree s in the $\mathrm{d}\bar{z}^k$'s, (z^k) being a system of local holomorphic coordinates on M. In this case d is of course the exterior differential and the standard notation is ∂ and $\bar{\partial}$ for $\mathrm{d}^{1,0}$ and $\mathrm{d}^{0,1}$. Furthermore if M is only an almost complex manifold, then it is true that $\Omega(M)$ is still a bigraded algebra but $\mathrm{d} = \partial + \bar{\partial}$ is true if and only if the almost complex structure is integrable [19]. So, among the almost complex manifolds, the complex manifolds are distinguished by the fact that their spaces of complex-valued differential forms are bigraded differential algebras.

In the following, we shall be interested on bigraded commutative algebras $\mathscr{A}$. Notice that $\mathscr{A}^{*,0}$ equipped with $\mathrm{d}^{1,0}$ is then a graded commutative algebra.

In the examples that we have in mind, $\mathscr{A}^{*,0}$ will be connected to differential forms and $d^{1,0}$ will be connected to the exterior differential; it is why we shall change a bit the notation by writing d to denote $d^{1,0}$ and δ to denote $d^{0,1}$, the total differential being now $d + \delta$. Thus, from now on, a bigraded differential algebra will be a bigraded algebra equipped with two anticommuting differentials d and δ such that $d\mathscr{A}^{r,s} \subset \mathscr{A}^{r+1,s}$ and $\delta\mathscr{A}^{r,s} \subset \mathscr{A}^{r,s+1}$. (In most examples in the following the differential algebra $(\mathscr{A}^{0,*}, \delta)$ will be related with a cochain complex of a Lie algebra).

Let $\mathfrak{g}$ be a Lie algebra and $\mathscr{A}$ be a bigraded commutative differential algebra (with the above notation). Suppose that i is an operation of $\mathfrak{g}$ on $\mathscr{A}$ considered as a graded commutative differential algebra. We shall say that $\mathscr{A}$ is a *bigraded* $\mathfrak{g}$*-operation* if i_X is bihomogeneous of bidegree $(-1, 0)$ and L_X is bihomogeneous of bidegree $(0, 0)$ for $X \in \mathfrak{g}$. In this case, one has $L_X = i_X d + d i_X$ and $i_X\delta + \delta i_X = 0$, for $X \in \mathfrak{g}$, and $(\mathscr{A}^{*,0}, d)$ is stable by i_X and is a $\mathfrak{g}$-operation; in fact, a sub-$\mathfrak{g}$-operation of $\mathscr{A}$. Notice that, although $(\mathscr{A}^{*,0}\ d)$ and $(\mathscr{A}^{0,*}, \delta)$ play a symmetric role in the bigraded differential algebra $\mathscr{A}$, in the last definition a preference is given to $(\mathscr{A}^{*,0}, d)$.

2.2. B.R.S. algebras and operations

As explained in the introduction, the relevant cohomology for the problem of anomalous terms in gauge theory is the δ-cohomology modulo d of the bigraded commutative algebra $B^{*,*}$. This algebra, as well as the algebra $\tilde{B}^{*,*}$ have special properties and, as will be shown below, it is useful to formalize by introducing the following concept.

DEFINITION 3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A *B.R.S. algebra over* $\mathfrak{g}$ is a pair $(\mathscr{A}, \omega)$ where $\mathscr{A}$ is a bigraded commutative differential algebra and ω is an element of $\mathfrak{g} \otimes \mathscr{A}^1$, $(\mathscr{A}^1 = \mathscr{A}^{1,0} \oplus \mathscr{A}^{0,1})$, such that we have

$$(*) \qquad (d + \delta)\omega + \frac{1}{2}[\omega, \omega] \in \mathfrak{g} \otimes \mathscr{A}^{2,0}$$

where d, δ and the bracket are defined on $\mathfrak{g} \otimes \mathscr{A}$, as before, by $dX \otimes \alpha = X \otimes d\alpha$, $\delta X \otimes \alpha = X \otimes \delta\alpha$ and $[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes \alpha\beta$ for $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathscr{A}$. ∎

One has $\omega = A + \chi$ with $A \in \mathfrak{g} \otimes \mathscr{A}^{1,0}$ and $\chi \in \mathfrak{g} \otimes \mathscr{A}^{0,1}$, so the above condition (*) reads

$$(d + \delta)(A + \chi) + \frac{1}{2}[A + \chi, A + \chi] = dA + \frac{1}{2}[A, A]$$

which is the condition (*) of the introduction so $\mathcal{B}^{*,*}$ and $\tilde{\mathcal{B}}^{*,*}$ are B.R.S. algebras over Lie (G). By expanding the above equation in bidegrees one obtains

$$(*\text{-}1) \qquad \delta A = -\,\mathrm{d}\chi - [A, \chi]$$

$$(*\text{-}2) \qquad \delta\chi = -\,\frac{1}{2}\,[\chi, \chi]$$

which is another form of (*).

When there is no ambiguity on $\omega = A + \chi$ we simply speak of the B.R.S. algebra $\mathscr{A}$; $\omega = A + \chi$ will be called the *algebraic connection*, or simply *the connection of* $\mathscr{A}$. In fact, for all the B.R.S. algebra $(\mathscr{A}, A + \chi)$ over $\mathfrak{g}$ that we have in mind, there is a natural bigraded $\mathfrak{g}$-operation for which $A + \chi$ is a connection (in the sense of 1.3). A B.R.S. algebra $(\mathscr{A}, A + \chi)$ over $\mathfrak{g}$ on which there is a bigraded $\mathfrak{g}$-operation for which $A + \chi$ is a connection will be called a *B.R.S. $\mathfrak{g}$-operation*. If $\mathscr{A}$ is a B.R.S. $\mathfrak{g}$-operation with connection $A + \chi$, then $\mathscr{A}^{*,0}$ is stable by the i_X, for $X \in \mathfrak{g}$, and the graded commutative differential algebra $(\mathscr{A}^{*,0}, \mathrm{d})$ is a $\mathfrak{g}$-operation with connection A.

Suppose that $\mathscr{A}$ is a B.R.S. $\mathfrak{g}$-operation with connection $A + \chi$, then one has (from the definitions) for any $Y \in \mathfrak{g}$

$$(4) \qquad \begin{aligned} &i_Y(A) = Y, && i_Y(\chi) = 0 \\ &i_Y(\mathrm{d}A) = [A, Y], && i_Y(\mathrm{d}\chi) = [\chi, Y] \end{aligned}$$

and of course $i_Y \delta + \delta i_Y = 0$.

It follows that we have, for $Y \in \mathfrak{g}$

$$L_Y \psi = [\psi, Y], \ \text{ for } \ \psi = A, \ \ \psi = \mathrm{d}A \ \ (\text{or } F), \ \ \psi = \chi$$

and $\psi = \mathrm{d}\chi$ and that $L_Y = i_Y \mathrm{d} + \mathrm{d} i_Y = i_Y(\mathrm{d} + \delta) + (\mathrm{d} + \delta) i_Y$ commutes with d and δ. In these formulae, i_Y and L_Y are again defined on $\mathfrak{g} \otimes \mathscr{A}$ by $i_Y(X \otimes \alpha) = X \otimes i_Y(\alpha)$ and $L_Y(X \otimes \alpha) = X \otimes L_Y(\alpha)$ for $X \in \mathfrak{g}$ and $\alpha \in \mathscr{A}$.

In the case of the algebra $\mathcal{B}^{*,*}$ with A and χ as in the introduction, formulae (4) define a bigraded Lie (G)-operation and $\mathcal{B}^{*,*}$ becomes a B.R.S. Lie (G)-operation.

Let P be a principal bundle over M with structure group G, $\dim M = n \in \mathbb{N}$ and G is a finite dimensional connected Lie group. The *gauge group of* P $\mathrm{Aut}_M(P)$ is the group of automorphisms of P living M (pointwise) invariant; i.e. $\zeta \in \mathrm{Aut}_M(P)$ if $\zeta(rg) = \zeta(r)g$ and $\pi(\zeta(r)) = \pi(r)$ for any $r \in P$ and $g \in G$, π being the projection $\pi : P \to M$. Introducing $g(r)$ such that $\zeta(r) = r\,g(r)$, for $\zeta \in \mathrm{Aut}_M(P)$, $r \in P$, one sees that $\mathrm{Aut}_M(P)$ identifies with the group of (smooth) mappings $g : P \to G$ such that $g(rg_0) = g_0^{-1}\, g(r) g_0$ for any $r \in P$, $g_0 \in G$. The Lie algebra $\mathrm{aut}_M(P)$

identifies with the mapping $\xi : P \to \mathrm{Lie}(G)$ such that $\xi(rg_0) = g_0^{-1}\xi(r)g_0$; in other words $\mathrm{aut}_M(P)$ identifies with the $\xi \in \mathrm{Lie}(G) \otimes \Omega^0(P)$ such that $L_Y \xi = [\xi, Y]$ $Y \in \mathrm{Lie}(G)$, where i_Y, L_Y are defined on $\Omega(P)$ and $\mathrm{Lie}(G) \otimes \Omega(P)$ as explained in 1.3 and the bracket on $\mathrm{Lie}(G) \otimes \Omega(P)$ is also as in 1.3. With the same notations, the (affine) space $\mathcal{C}$ of connections on P is the space of all $\alpha \in \mathrm{Lie}(G) \otimes \Omega^1(P)$ satisfying $i_Y(\alpha) = Y$ and $L_Y \alpha = [\alpha, Y]$, for $Y \in \mathrm{Lie}(G)$. Let $\widetilde{\mathcal{B}}^{r,s}(P)$ be the space of differential operators of $\mathcal{C} \times (\mathrm{aut}_M(P))^s$ in $\Omega^r(P)$ which are polynomial in $\mathcal{C}$ and s-linear antisymmetric in $(\mathrm{aut}_M(P))^s$ and let $\widetilde{\mathcal{B}}^{*,*}(P) = \oplus_{r,s} \widetilde{\mathcal{B}}^{r,s}(P)$. By the same formulae as for $\widetilde{\mathcal{B}}^{*,*}$ in the introduction, one defines a product on $\widetilde{\mathcal{B}}^{*,*}(P)$ for which $\widetilde{\mathcal{B}}^{*,*}(P)$ is a bigraded commutative algebra and two differentials d and δ for which it becomes a bigraded commutative differential algebra; d is induced by the exterior differential of $\Omega(P)$ and δ is induced by the Lie algebra structure of $\mathrm{aut}_M(P)$ and the right action of $\mathrm{aut}_M(P)$ on $\mathcal{C}$. On $\Omega(P)$ we have already defined i_X and L_X for $X \in \mathrm{Lie}(G)$, (see in 1.3); we extend i_X and L_X to $\widetilde{\mathcal{B}}^{*,*}(P)$ by writing (for $X \in \mathrm{Lie}(G)$)

$$(i_X \omega)(\alpha; \xi_1, \ldots, \xi_x) = i_X(\omega(\alpha; \xi_1, \ldots, \xi_s)),$$

and

$$(L_X \omega)(\alpha; \xi_1, \ldots, \xi_x) = L_X(\omega(\alpha; \xi_1, \ldots, \xi_s)),$$

for $\omega \in \widetilde{\mathcal{B}}^{*,*}(P)$, $\alpha \in \mathcal{C}$ and $\xi_1, \ldots, \xi_s \in \mathrm{aut}_M(P)$. One verifies that, with this $X \mapsto i_X$, $\mathcal{B}^{*,*}(P)$ is a bigraded Lie (G)-operation. Furthermore, by defining $A \in \mathrm{Lie}(G) \otimes \widetilde{\mathcal{B}}^{1,0}(p)$ by $A(\alpha) = \alpha \in \mathrm{Lie}(G) \otimes \Omega^1(P)$ for $\alpha \in \mathcal{C}$ and $\chi \in \mathrm{Lie}(G) \otimes \widetilde{\mathcal{B}}^{0,1}(P)$ by $\chi(\xi) = \xi \in \mathrm{Lie}(G) \otimes \Omega^0(P)$ for $\xi \in \mathrm{aut}_M(P)$, one verifies that $A + \chi$ is a connection on the bigraded Lie (G)-operation $\widetilde{\mathcal{B}}^{*,*}(P)$ for which the condition (*) is verified. Thus $\widetilde{\mathcal{B}}^{*,*}(P)$ is a B.R.S. Lie (G)-operation and $\widetilde{\mathcal{B}}^{*,0}(P)$ is just the Lie (G)-operation with connection A defined at the end of 1.3.

Let $\mathcal{B}^{*,*}(P)$ be the smallest bigraded differential subalgebra (with unit) of $\widetilde{\mathcal{B}}^{*,*}(P)$ wich contains the Lie (G)-components of A and χ; i.e. if (E_α) is a basis of $\mathrm{Lie}(G)$ and if $A = E_\alpha \otimes A^\alpha$ and $\chi = E_\alpha \otimes \chi^\alpha$, $\mathcal{B}^{*,*}(P)$ is the subalgebra of $\widetilde{\mathcal{B}}^{*,*}(P)$ generated by the A^α's the χ^α's, the $\mathrm{d}A^\alpha$'s and the $\mathrm{d}\chi^\alpha$'s (the δA^α's and $\delta\chi^\alpha$'s are then in $\mathcal{B}^{*,*}(P)$ by (*)). $\mathcal{B}^{*,*}(P)$ is a bi-graded differential subalgebra of $\widetilde{\mathcal{B}}^{*,*}(P)$ which is stable by the i_X, for $X \in \mathrm{Lie}(G)$ and, by construction, $A + \chi \in \mathrm{Lie}(G) \otimes \mathcal{B}^{*,*}(P)$. It follows that $\mathcal{B}^{*,*}(P)$ is a B.R.S. Lie (G)-operation (in fact a sub-operation of $\widetilde{\mathcal{B}}^{*,*}(P)$). Furthermore, $\mathcal{B}^{*,0}(P)$ is the Lie (G)-operation with connection A which was also introduced at the end of 1.3.

In 1.3 we constructed a $\mathfrak{g}$-operation with connection $\mathcal{B}^{*,0}(\mathscr{A})$ for any $\mathfrak{g}$-operation $\mathscr{A}$ which admits connections and $\mathcal{B}^{*,0}(P) = \mathcal{B}^{*,0}(\Omega(P))$ was the particular case where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathscr{A} = \Omega(P)$. In order to show that the construction of $\mathcal{B}^{*,*}(P)$ is really a natural one and in order to get some insight on what generalises the notion of gauge transformation for the case of a $\mathfrak{g}$-operation $\mathscr{A}$, we shall proceed to the construction of a B.R.S. $\mathfrak{g}$-operation $B^{*,*}(\mathscr{A})$ associated to a $\mathfrak{g}$-operation $\mathscr{A}$ which admits connections. $B^{*,*}(P)$ will appear as the particular case where $\mathfrak{g} = \mathrm{Lie}\,(G)$ and $\mathscr{A} = \Omega(P)$ of this construction, i.e. $B^{*,*}(P) = B^{*,*}(\Omega(P))$.

First of all, let us say a few words on the structure of the graded derivations of a graded commutative algebra $\mathscr{A}$. We denote by $D^k(\mathscr{A})$ the space of graded derivations of degree $k \in \mathbb{Z}$ of $\mathscr{A}$; if k is even the elements of $D^k(\mathscr{A})$ are derivations and if k is odd the elements of $D^k(\mathscr{A})$ are antiderivations. As already noticed in 1.1 the $\mathbb{Z}$-graded space $D(\mathscr{A}) = \oplus_k D^k(\mathscr{A})$ is a $\mathbb{Z}$-graded Lie algebra for the graded commutator $[\cdot\,,\cdot]$ defined by

$$[\delta, \delta'](\omega) = \delta(\delta'(\omega)) - (-1)^{kk'}\delta'(\delta(\omega))$$

for $\delta \in D^k(\mathscr{A})$, $\delta' \in D^{k'}(\mathscr{A})$ and $\omega \in \mathscr{A}$. On the other hand $D(\mathscr{A})$ is also a graded $\mathscr{A}$-module if we define $\alpha\delta$ for $\alpha \in \mathscr{A}$ and $\delta \in D(\mathscr{A})$ by $(\alpha\delta)(\omega) = \alpha\delta(\omega)$ for any $\omega \in \mathscr{A}$; one verifies easily, by using the graded commutativity, that if $\alpha \in \mathscr{A}^m$ and $\delta \in D^n(\mathscr{A})$ then $\alpha\delta \in D^{m+n}(\mathscr{A})$. It is worth noticing here that although $D(\mathscr{A})$ is a graded Lie algebra and a graded $\mathscr{A}$-module, *it is not a graded Lie algebra over* $\mathscr{A}$ since the graded commutator $[\cdot\,,\cdot]$ is not $\mathscr{A}$-bilinear. All this of course applies in the case of a $\mathbb{Z}$-graded commutative algebra $\mathscr{A} = \oplus_{k\in\mathbb{Z}} \mathscr{A}^k$, but we shall be interested only in the case where $\mathscr{A}^k = \{0\}$ for $k < 0$, i.e. is the case where $\mathscr{A}$ in $\mathbb{N}$-graded. Notice also that a commutative algebra $\mathbf{A}$ is a graded commutative algebra if we define $\mathbf{A}^0 = \mathbf{A}$ and $\mathbf{A}^k = \{0\}$ for $k \neq 0$; then $D(\mathbf{A}) = D^0(\mathbf{A})$ is the Lie algebra of derivations of $\mathbf{A}$. This applies in particular to the algebra $C^\infty(M)$ of smooth functions on a manifold M where $D(C^\infty(M)) = D^0(C^\infty(M))$ is the space of vector fields on M which is, as well known, a Lie algebra and a $C^\infty(M)$-module but not a Lie algebra over $C^\infty(M)$.

Let us now assume that $\mathscr{A}$ is a $\mathfrak{g}$-operation. When $\mathfrak{g} = \mathrm{Lie}\,(G)$ and $\mathscr{A} = \Omega(P)$ an automorphism of P is simply an automorphism of the Lie (G)-operation $\Omega(P)$; the set of automorphisms of P, $\mathrm{Aut}\,(P)$ is a group, and the gauge group $\mathrm{Aut}_M(P)$ of P identifies with the subgroup of the elements of $\mathrm{Aut}\,(P)$ which leave invariant each basic element of $\Omega(P)$. This generalized immediately to $(\mathscr{A}, \mathfrak{g})$. We call *gauge transformation of the* $\mathfrak{g}$*-operation* $\mathscr{A}$ any automorphism of the $\mathfrak{g}$-operation $\mathscr{A}$ which leaves invariant each element of $\mathscr{B}(\mathscr{A})$; these gauge transformations form a group denoted by $\mathrm{Aut}_{\mathscr{B}}(\mathscr{A})$, (thus $\mathrm{Aut}_M(P)$ identifies with $\mathrm{Aut}_{\mathscr{B}}(\Omega(P))$). At the infinitesimal level, we denote by $\mathrm{aut}_{\mathscr{B}}(\mathscr{A})$, the Lie algebra of derivations θ of degree zero of $\mathscr{A}$ which commute with the i_X ($X \in \mathfrak{g}$), $\theta i_X = i_X\theta$, and are such that $\theta\beta = 0$ for any $\beta \in \mathscr{B}(\mathscr{A})$ (i.e. for any $\beta \in \mathscr{A}$ such that $i_Y\beta = 0$ and $L_Y\beta = 0$ for any $Y \in \mathfrak{g}$). In the case $\mathfrak{g} = \mathrm{Lie}\,(G)$ and

$\mathscr{A} = \Omega(P)$, $\mathrm{aut}_{\mathscr{B}}(\mathscr{A}) = \mathrm{aut}_M(P)$ identifies with the Lie subalgebra of $\mathrm{Lie}\,(G) \otimes \Omega^0(P)$ which consists of $\xi \in \mathrm{Lie}\,(G) \otimes \Omega^0(P)$ satisfying $L_Y \xi = [\xi, Y]$, $\forall Y \in \mathrm{Lie}\,(G)$. In the general case $(\mathscr{A}, \mathfrak{g})$, let us denote by $\mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$ the Lie subalgebra of $\mathfrak{g} \otimes \mathscr{A}^0$ of the $\xi \in \mathfrak{g} \otimes \mathscr{A}^0$ satisfying $L_Y \xi = [\xi, Y]$, $\forall Y \in \mathfrak{g}$; we shall define a Lie algebra homomorphism $L : \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A}) \to \mathrm{aut}_{\mathscr{B}}(\mathscr{A})$. Let (E_α) be an arbitrary basis of $\mathfrak{g}$, $\xi = \xi^\alpha \otimes E_\alpha$ be an element of $\mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$ and define $L_\xi(\omega) = \mathrm{d}(\xi^\alpha i_{E_\alpha}(\omega)) + \xi^\alpha i_{E_\alpha}(\mathrm{d}\omega)$ for $\omega \in \mathscr{A}$; then $L_\xi(i_Y(\omega)) = i_Y L_\xi(\omega)$ is easily verified and, since $L_\xi(\omega) = \xi^\alpha L_{E_\alpha}(\omega) + (\mathrm{d}\xi^\alpha) i_{E_\alpha}(\omega)$, it follows that $L_\xi(\omega) = 0$ whenever ω is basic. On the other hand it follows from the above considerations that L_ξ is in $D^0(\mathscr{A})$ and that L defines in fact a Lie algebra homomorphism $L : \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A}) \to \mathrm{aut}_{\mathscr{B}}(\mathscr{A})$.

Let $\mathscr{A}$ be a $\mathfrak{g}$-operation which admits connections and let C denote the affine space of connections on $\mathscr{A}$. Consider the space $P^{r,s}(\mathscr{A})$ of mapping of $C \times (\mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A}))^s$ in $\mathscr{A}^r$ which are polynomial in C and s-linear antisymmetric in $\mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$. One defines on $\underset{r,s}{\oplus} P^{r,s}(\mathscr{A}) = P(\mathscr{A})$ a product for which it becomes a bigraded commutative algebra and differential d and δ for which $P(\mathscr{A})$ is a bigraded commutative differential algebra by the same formulae as in the introduction, namely

$$(\omega \cdot \omega')(a; \xi_1, \ldots, \xi_{s+s'}) =$$

$$= \frac{(-1)^{r's}}{(s+s')!} \sum_{\pi \in \mathfrak{S}_{s+s'}} (-1)^{\epsilon(\pi)} \omega(a; \xi_{\pi(1)}, \ldots, \xi_{\pi(s)}) \omega'(a; \xi_{\pi(s+1)}, \ldots, \xi_{\pi(s+s')})$$

for $\omega \in P^{r,s}(\mathscr{A})$, $\omega' \in P^{r',s'}(\mathscr{A})$, $a \in C$ and $\xi_i \in \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$, $(\mathrm{d}\omega)(a; \xi_1, \ldots, \xi_s) = \mathrm{d}(\omega(a; \xi_1, \ldots, \xi_s))$ and

$$(\delta\omega)(a; \xi_0, \ldots, \xi_s) = \sum_{0 \leq k \leq s} (-1)^{r+k} \theta(\xi_k) \omega(a; \xi_0, \ldots, \hat{\xi}_k, \ldots, \xi_s) +$$

$$+ \sum_{0 \leq l < m \leq s} (-1)^{r+l+m} \omega(a; [\xi_l, \xi_m], \xi_0, \ldots, \hat{\xi}_l, \ldots, \hat{\xi}_m, \ldots, \xi_s)$$

for $\omega \in P^{r,s}(\mathscr{A})$ $a \in C$ and $\xi_i \in \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$; where $\theta(\xi)$ is induced, for $\xi \in \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$, by the right action on C $a \mapsto L_\xi a = \mathrm{d}\xi + [a, \xi]$. Introducing $A \in \mathfrak{g} \otimes P^{1,0}(\mathscr{A})$ and $\chi \in \mathfrak{g} \otimes P^{0,1}(\mathscr{A})$ by $A(a) = a \in \mathfrak{g} \otimes \mathscr{A}^1$ and $\chi(\xi) = \xi \in \mathfrak{g} \otimes \mathscr{A}^0$ for $a \in C$ and $\xi \in \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A})$, one readily verifies that $P(\mathscr{A})$ is a B.R.S. algebra over $\mathfrak{g}$ with connection $A + \chi$ and that furthermore, if we define i_X on $P(\mathscr{A})$ for $X \in \mathfrak{g}$ by $(i_X \omega)(a; \xi_1, \ldots, \xi_s) = i_X(\omega(a; \xi_1, \ldots, \xi_s))$, $(\omega \in P^{r,s}(\mathscr{A})$, $a \in C$ and $\xi_i \in \mathrm{aut}^{(0)}_{\mathscr{B}}(\mathscr{A}))$, $P(\mathscr{A})$ becomes a B.R.S. $\mathfrak{g}$-operation. One then defines

$\mathcal{B}^{*,*}(\mathscr{A})$ to be the smallest bigraded differential subalgebra (with unit) of $P(\mathscr{A})$ which contains the components (in $\mathfrak{g}$) of A and χ; $\mathcal{B}^{*,*}(\mathscr{A})$ is again a B.R.S. $\mathfrak{g}$-operation and, in the above case $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathscr{A} = \Omega(P)$ it reduces to $\mathcal{B}^{*,*}(P)$. $\mathcal{B}^{*,0}(\mathscr{A})$ is of course the $\mathfrak{g}$-operation with connection A introduced in 1.3.

2.3. The Weil - B.R.S. algebra of a Lie algebra

Given a finite dimensional Lie algebra $\mathfrak{g}$, there is a natural notion of homomorphism of B.R.S. algebra over $\mathfrak{g}$, namely it is a homomorphism of the corresponding bigraded differential algebras mapping the connection on the connection; in the same way there is a natural notion of homomorphism of B.R.S. $\mathfrak{g}$-operation, namely it is homomorphism of the underlying B.R.S. algebras which is also a homomorphism of $\mathfrak{g}$-operations. It turns out that in the category of B.R.S.-algebras there is a universal initial object which we call the Weil - B.R.S. algebra of $\mathfrak{g}$ and which we denote by $\mathsf{A}(\mathfrak{g})$. Furthermore there is a natural $\mathfrak{g}$-operation on $\mathsf{A}(\mathfrak{g})$ for which $\mathsf{A}(\mathfrak{g})$ is a B.R.S. $\mathfrak{g}$-operation and it turns out that, as B.R.S. $\mathfrak{g}$-operation, $\mathsf{A}(\mathfrak{g})$ is also a universal initial object in the category of B.R.S. $\mathfrak{g}$-operations. We now describe $\mathsf{A}(\mathfrak{g})$.

As graded algebra, $\mathsf{A}(\mathfrak{g})$ is described by

$$\mathsf{A}(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$, $\otimes$ is the tensor product of graded algebra and $S\mathfrak{g}^*$ is, as before, considered as a graded commutative algebra by giving the degree $2k$ to the elements of $S^k\mathfrak{g}^*$ ($S^k\mathfrak{g}^* = (S\mathfrak{g}^*)^{2k}$). Let (E_α) be a basis of $\mathfrak{g}$ with dual basis (E^α) and define A^α, F^α, χ^α and ψ^α in $\mathsf{A}(\mathfrak{g})$ by $A^\alpha = E^\alpha \otimes \mathbb{1} \otimes \otimes \mathbb{1} \otimes \mathbb{1}$, $F^\alpha = \mathbb{1} \otimes E^\alpha \otimes \mathbb{1} \otimes \mathbb{1}$, $\chi^\alpha = \mathbb{1} \otimes \mathbb{1} \otimes E^\alpha \otimes \mathbb{1}$ and $\psi^\alpha = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes E^\alpha$. $\mathsf{A}(\mathfrak{g})$ is just the free connected graded commutative algebra generated by the $A^\alpha{}'s$ and the $\chi^\alpha{}'s$ in degree one and by the $F^\alpha{}'s$ and the $\psi^\alpha{}'s$ in degree two. Introducing A, F, χ and ψ in $\mathfrak{g} \otimes \mathsf{A}(\mathfrak{g})$ by $A = E_\alpha \otimes A^\alpha$, $F = E_\alpha \otimes F^\alpha$, $\chi = E_\alpha \otimes \chi^\alpha$ and $\psi = E_\alpha \otimes \psi^\alpha$, we define $\mathrm{d}A^\alpha$, $\mathrm{d}F^\alpha$, $\mathrm{d}\chi^\alpha$, $\mathrm{d}\psi^\alpha$, δA^α, δF^α, $\delta\chi^\alpha$ and $\delta\psi^\alpha$ in $\mathsf{A}(\mathfrak{g})$ by $\mathrm{d}A = -\frac{1}{2}[A, A] + F$, $\mathrm{d}F = -[A, F]$, $\mathrm{d}\chi = \psi$, $\mathrm{d}\psi = 0$ and $\delta A = -\psi - [A, \chi]$, $\delta F = [F, \chi]$, $\delta\chi = -\frac{1}{2}[\chi, \chi]$, $\delta\psi = [\psi, \chi]$, where $\mathrm{d}A = E_\alpha \otimes \mathrm{d}A^\alpha, \ldots, \delta\psi = E_\alpha \otimes \delta\psi^\alpha$. d and δ extend uniquely as two antiderivations of $\mathsf{A}(\mathfrak{g})$ and one verifies that d and δ so defined are two anticommuting differential, i.e. they are of degree one and satisfy $\mathrm{d}^2 = 0$, $\delta^2 = 0$ and $\mathrm{d}\delta + \delta\mathrm{d} = 0$. One then introduces an underlying bigraduation on $\mathsf{A}(\mathfrak{g})$, for which it is a bigraded commutative algebra, by giving to the $A^\alpha{}'s$ the bidegree $(1, 0)$, to the $F^\alpha{}'s$ the bidegree $(2, 0)$, to the $\chi^\alpha{}'s$ the bidegree $(0, 1)$ and to the $\psi^\alpha{}'s$ the bidegree $(1, 1)$. Then d and δ are homogeneous for the bidegree of respective bidegrees $(1, 0)$ and $(0, 1)$, so $\mathsf{A}(\mathfrak{g})$ is a bigraded commutative differential algebra. It follows from the definition that we have

$$F = \mathrm{d}A + \frac{1}{2}[A, A] = (\mathrm{d} + \delta)(A + \chi) + \frac{1}{2}[A + \chi, A + \chi],$$

thus, $\mathsf{A}(\mathfrak{g})$ is a B.R.S. algebra over $\mathfrak{g}$ with connection $A + \chi$. Let us define i_X for $X \in \mathfrak{g}$ to be the unique antiderivation of $\mathsf{A}(\mathfrak{g})$ satisfying $i_X(A) = X$, $i_X(F) = 0 = i_X(\chi)$ and $i_X(\psi) = [\chi, X]$ with, as usual here,

$$i_X(A) = E_\alpha \otimes i_X(A^\alpha), \ldots, i_X(\psi) = E_\alpha \otimes i_X(\psi^\alpha)$$

and the standard bracket on $\mathfrak{g} \otimes \mathsf{A}(\mathfrak{g})$. Equipped with $X \mapsto i_X$, $\mathsf{A}(\mathfrak{g})$ becomes a B.R.S. $\mathfrak{g}$-operation, as easily verified, and we shall refer to it as *the Weil - B.R.S. algebra of the Lie algebra* $\mathfrak{g}$. One has the following theorem.

THEOREM 8. (Universal property of $\mathsf{A}(\mathfrak{g})$). *For any B.R.S. algebra $\mathscr{A}$ over $\mathfrak{g}$, there is a unique homomorphism of B.R.S. algebras over $\mathfrak{g}$, $\mathfrak{f} : \mathsf{A}(\mathfrak{g}) \to \mathscr{A}$, of $\mathsf{A}(\mathfrak{g})$ in $\mathscr{A}$. If furthermore, $\mathscr{A}$ is a B.R.S. $\mathfrak{g}$-operation, then $\mathfrak{f}$ is an homomorphism of B.R.S. $\mathfrak{g}$-operations.*

The proof of this theorem is completely similar to the proof of theorem 2 and just use the universal properties of exterior algebras, symmetric algebras and tensor products. ∎

Notice that $\mathsf{A}^{*,0}(\mathfrak{g}) = \bigoplus_r \mathsf{A}^{r,0}(\mathfrak{g})$ is isomorphic to the Weil algebra; in fact $\mathsf{A}^{*,0}(\mathfrak{g}) = W(\mathfrak{g}) \otimes \mathbb{1} \otimes \mathbb{1}$. More generally, if we retain only the total degree, it follows from $(\mathrm{d} + t\delta)(A + t\chi) + \frac{1}{2}[A + t\chi, A + t\chi] = F$, with $t \in \mathbb{R}$, that the subalgebra $W_t(\mathfrak{g})$ of $\mathsf{A}(\mathfrak{g})$ generated by the $A^\alpha + t\chi^\alpha$'s and the F^α's is stable by $\mathrm{d}_t = \mathrm{d} + t\delta$ and by the i_X $(X \in \mathfrak{g})$ and that it is isomorphic to the Weil algebra with differential d_t and connection $A_t = A + t\chi$. Thus $\mathsf{A}(\mathfrak{g})$ is a sort of «δ-dressing» of $W(\mathfrak{g})$; in the same way, a B.R.S. $\mathfrak{g}$-operation $\mathscr{A}$ appears as a «δ-dressing» of the $\mathfrak{g}$-operation $\mathscr{A}^{*,0}$ although $\mathscr{A}$ is itself a $\mathfrak{g}$-operation of a particular type.

We shall need later on the d-cohomology, the $(\mathrm{d} + \delta)$-cohomology and the δ-cohomology of $\mathsf{A}(\mathfrak{g})$ which we now describe.

Let us denote by $\mathscr{T}_{\mathsf{A}}(\mathfrak{g})$ the subalgebra of invariant elements of $\mathsf{A}(\mathfrak{g})$, (i.e. $\mathscr{T}_{\mathsf{A}}(\mathfrak{g}) = \mathscr{T}(\mathsf{A}(\mathfrak{g}))$) with the notations of 1.3). Then $\mathscr{T}_{\mathsf{A}}(\mathfrak{g})$ is stable by d and by δ since L_X commutes with d and δ for any $X \in \mathfrak{g}$.

THEOREM 9. *The* d-*cohomologies and the* $(\mathrm{d} + \delta)$-*cohomologies of* $\mathsf{A}(\mathfrak{g})$ *and of the subalgebra* $\mathscr{T}_{\mathsf{A}}(\mathfrak{g})$ *are trivial.*

I.e. they all reduce to the ground field $\mathbb{K}$ (in degree zero). The proof is exactly the same as the one of theorem 5. In fact, one notices that for any $t \in \mathbb{R}$, A^α, $(\mathrm{d} + t\delta) A^\alpha$ χ^α and $(\mathrm{d} + t\delta)\chi^\alpha$ is a free system of homogeneous generators of $\mathrm{A}(\mathfrak{g})$ and that the unique antiderivation k_t satisfying $k_t A^\alpha = k_t \chi^\alpha = 0$ and $k_t(\mathrm{d} + t\delta)A^\alpha = A^\alpha$, $k_t(\mathrm{d} + t\delta)\chi^\alpha = \chi^\alpha$ commutes with the L_X and satisfies $k_t(\mathrm{d} + t\delta) + (\mathrm{d} + t\delta)k_t =$ «degree in the generators $\{A^\alpha, \chi^\alpha, (\mathrm{d} + t\delta) A^\alpha, (\mathrm{d} + t\delta)\chi^\alpha\}$» so it gives a contracting homotopy in positive degrees for both $\mathrm{A}(\mathfrak{g})$ and $\mathscr{T}_{\mathrm{A}}(\mathfrak{g})$. So the $(\mathrm{d} + t\delta)$-cohomologies of both $\mathrm{A}(\mathfrak{g})$ and $\mathscr{T}_{\mathrm{A}}(\mathfrak{g})$ are trivial for any $t \in \mathbb{R}$. ∎

THEOREM 10. *The δ-cohomology of $\mathrm{A}(\mathfrak{g})$ coincides with the δ-cohomology of the subalgebra (with unit) generated by the χ^α, F^α; this differential subalgebra of $\mathrm{A}(\mathfrak{g})$ for δ identifies with the complex $C^*(\mathfrak{g}, S\mathfrak{g}^*)$ of cochains on $\mathfrak{g}$ with values in $S\mathfrak{g}^*$. So one has $H^{2k+1,s}(\mathrm{A}(\mathfrak{g}), \delta) = 0$ and $H^{2k,s}(\mathrm{A}(\mathfrak{g}), \delta) = H^s(\mathfrak{g}, S^k\mathfrak{g}^*)$ for any $k, s \in \mathbb{N}$.*

For the proof of this theorem, one notices that the A^α, δA^α, χ^α and F^α form a free system of bihomogeneous generators of $\mathrm{A}(\mathfrak{g})$ and that the subalgebra generated by the χ^α and the F^α is stable by δ and identifies with $C^*(\mathfrak{g}, S\mathfrak{g}^*) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ and that δ is just the differential of $C^*(\mathfrak{g}, S\mathfrak{g}^*)$, (see in 1.2d and in the «warning» of 1.4). Thus $(\mathrm{A}(\mathfrak{g}), \delta)$ is the tensor product of the contractible algebra $\underset{\alpha}{\otimes} \mathscr{C}(A^\alpha, \delta A^\alpha)$ with $C^*(\mathfrak{g}, S\mathfrak{g}^*)$, i.e. $(\mathrm{A}(\mathfrak{g}), \delta) = (\underset{\alpha}{\otimes} \mathscr{C}(A^\alpha, \delta A^\alpha)) \otimes C^*(\mathfrak{g}, S\mathfrak{g}^*)$, so $H(\mathrm{A}(\mathfrak{g}), \delta) = H^*(\mathfrak{g}, S\mathfrak{g}^*)$. The rest follows by the correct identification of the bidegree. ∎

The main reason here why we introduced $\mathrm{A}(\mathfrak{g})$ is the following extension of Lemma 1 for the B.R.S. algebra $\mathcal{B}^{*,*}$.

LEMMA 2. *The canonical homomorphism $\ell_{\mathcal{B}^{*,*}} : \mathrm{A}(\mathrm{Lie}\,(G)) \to \mathcal{B}^{*,*}$ of B.R.S. algebras over* Lie (G) *is surjective and induces isomorphisms of vector spaces $\mathrm{A}^{r,s}(\mathrm{Lie}\,(G)) \to \mathcal{B}^{r,s}$ for any (r, s) with $r \leqslant \dim(M)$.*

The proof is similar to the proof of lemma 1 and, in fact, lemma 1 is the «hard part» of this theorem. ∎

One has (by definition) a surjective homomorphism $\mathcal{B}^{*,*}(M \times G) \to \mathcal{B}^{*,*}$ of B.R.S. algebras over Lie (G), obtained by restriction to the canonical section $M \times \mathbb{1}$ of the trivial G-principal bundle $M \times G$. As explained in 2.2, $\mathcal{B}^{*,*}(M \times G)$ is not only a B.R.S. algebra but also a B.R.S. Lie (G)-operation. Now although the i_X for $X \in \mathrm{Lie}\,(G)$, are not well defined on $\mathcal{B}^{*,*}$, the L_X are perfectly defined on $\mathcal{B}^{*,*}$ and the homomorphism $\mathcal{B}^{*,*}(M \times G) \to \mathcal{B}^{*,*}$ permutes with the actions of the L_X on $\mathcal{B}^{*,*}(M \times G)$ and on $\mathcal{B}^{*,*}$, for $X \in \mathrm{Lie}\,(G)$. It follows then from the last lemma that $\mathfrak{f}_{\mathcal{B}^{*,*}}$ permutes with the actions of the L_X on $\mathcal{A}(\mathrm{Lie}\,(G))$ and $\mathcal{B}^{*,*}$, since it has to factorize through $\mathfrak{f}_{\mathcal{B}^{*,*}(M \times G)}$ in view of the universal property of $\mathcal{A}(\mathfrak{g})$, (in fact one could as well define L_X on $\mathcal{B}^{*,*}$ by this property of $\mathfrak{f}_{\mathcal{B}^{*,*}}$). In particular the invariant elements of $\mathcal{A}(\mathrm{Lie}\,(G))$ are mapped by $\mathfrak{f}_{\mathcal{B}^{*,*}}$ on the invariant elements of $\mathcal{B}^{*,*}$.

It follows from the last lemma that the δ-cohomology modulo d of $\mathcal{B}^{*,*}$ is completely known from the one of $\mathcal{A}(\mathrm{Lie}\,(G))$. It is why the rest of this paper is completely devoted to the computation of the δ-cohomology modulo d of $\mathcal{A}(\mathfrak{g})$ and we shall end with a complete description of this cohomology in the case where $\mathfrak{g}$ is a reductive Lie algebra.

Again, by the same argument as the one leading from lemma 1 to theorem 7, one has the following extension of theorem 7.

THEOREM 11. *Let P be a G-principal bundle over M. Then the canonical homomorphism of* $\mathcal{A}(\mathrm{Lie}\,(G))$ *in* $\mathcal{B}^{*,*}(P)$ *(of B.R.S.* Lie (G)*-operations) is surjective and induces isomorphisms of vector spaces* $\mathcal{A}^{r,s}(\mathrm{Lie}\,(G)) \to \mathcal{B}^{r,s}(P)$ *for any* (r, s) *with* $r \leqslant \dim(M)$.

3. THE δ-COHOMOLOGY MODULO d OF THE WEIL-B.R.S. ALGEBRA

3.1. Complements on differential vector spaces

All the vector spaces considered here are vector spaces with the same ground field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. We denote by 0 the (0-dimensional) vector space which is reduced to its zero element; this vector space has the properties that, given an arbitrary vector V there is a unique linear mapping $0 \to V$ of 0 in V, (namely $0 \to 0 \in V$), and a unique linear mapping $V \to 0$, (namely $x \to 0$, $\forall x \in V$). In categorial language the first property means that 0 is an initial object in the category of vector spaces (over $\mathbb{K}$) and the second property means that 0 is a final object in the category of vector spaces.

A sequence $\dots V_{n-1} \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} \dots$ of vector spaces and linear mappings is said to be *exact at* V_n if $\mathrm{Im}\,(f_{n-1}) = \mathrm{Ker}\,(f_n)$, i.e. $f_{n-1}(V_{n-1}) = f_n^{-1}(0)$; it is said to be an *exact sequence* if it is exact everywhere, (for any n). For instance, the sequence $0 \to F \xrightarrow{i} G$ is exact if and only if i is injective ($i^{-1}(0) = 0$), the $G \xrightarrow{p} H \to 0$ is exact if and only if p is surjective ($p(G) = H$), so $0 \xrightarrow{f} V \to W \to 0$ is exact if and only if f is an isomorphism. A *shorth exact sequence* of vector spaces in an exact sequence of the form $0 \to F \xrightarrow{i} G \xrightarrow{p} H \to 0$; exactness at F

means that i is injective so F is isomorphic to the subspace $i(F)$ of G, exactness at H means that p is surjective so H is isomorphic to $G/p^{-1}(0)$ i.e. to $G/i(F)$ by exactness at G.

A *differential vector space*, or a *differential space*, is a vector space V equipped with an endomorphism d such that $\mathrm{d}^2 = 0$; d is called *the differential of* V. Given a differential space V, one defines the subspaces $B(V) = \mathrm{Im}\,(\mathrm{d}) = \mathrm{d}(V)$ and $Z(V) = \mathrm{Ker}\,(\mathrm{d}) = \{x \in V \mid \mathrm{d}x = 0\}$; one has, from $\mathrm{d}^2 = 0$, $B(V) \subset Z(V)$ and the vector space $H(V) = Z(V)/B(V)$ is the *homology* of the differential vector space V. If V' is another differential space with differential d$'$, an *homomorphism of differential spaces* of V in V' is a linear mapping $f : V \to V'$ such that $f \circ \mathrm{d} = \mathrm{d}' \circ f$. It follows from this definition that if $f : V \to V'$ is an homomorphism of differential space, one has $f(B(V)) \subset B(V')$ and $f(Z(V)) \subset Z(V')$ so f induces a linear mapping $f^{\#} : H(V) \to H(V')$ in homology. If $f' : V' \to V''$ is another homomorphism of differential spaces, then $f' \circ f : V \to V''$ is of course again an homomorphism of differential spaces and, one has in homology

$$(f' \circ f)^{\#} = f'^{\#} \circ f^{\#} \; : H(V) \to H(V'').$$

One has an obvious notion of exact sequence of differential spaces; this is just a sequence of homomorphism of differential spaces which is exact as sequence of linear mappings of vector spaces. Given an exact sequence of differential spaces, the corresponding sequence of linear mapping in homology is generally *not exact*. For instance if $0 \to E \xrightarrow{i} F \xrightarrow{p} G \to 0$ is a short exact sequence of differential spaces, one verifies easily that the sequence $H(E) \xrightarrow{i^{\#}} H(F) \xrightarrow{p^{\#}} H(G)$ is exact (at $H(F)$), however $i^{\#}$ is generally not injective and $p^{\#}$ is generally not surjective so the sequence $0 \to H(E) \xrightarrow{i^{\#}} H(F) \xrightarrow{p^{\#}} H(G) \to 0$ is generally not exact (except at $H(F)$). One has the following classical result:

PROPOSITION 1. *Given a short exact sequence of differential vector spaces* $0 \to$ $\to E \xrightarrow{i} F \xrightarrow{p} G \to 0$, *there is a linear mapping* $\partial : H(G) \to H(E)$ *for which the triangle*

$$\begin{tikzcd}
H(G) \arrow[rr, "\partial"] & & H(E) \arrow[dl, "i^{\#}"] \\
& H(F) \arrow[ul, "p^{\#}"] &
\end{tikzcd}$$

in exact. Furthermore ∂ *has the following (functorial) property: Given another short exact sequence of differential spaces* $0 \to E' \xrightarrow{i'} F' \xrightarrow{p'} G' \to 0$ *and homomorphism of differential spaces* $e : E \to E'$, $f : F \to F'$, $g : G \to G'$ *such that* $i' \circ e =$ $= f \circ i$ *and* $p' \circ f = g \circ p$, *one has* $e^{\#} \circ \partial = \partial \circ g^{\#}$.

Let us remind the definition of ∂. Let $h \in H(G)$ and let $x \in Z(G)$ be in the class h. By surjectivity of p (exactness at G) there is a $y \in F$ such that $p(y) = x \cdot p(\mathrm{d}y) = \mathrm{d}x = 0$ so (exactness at F) there is a $z \in E$ such that $i(z) = \mathrm{d}y$. One has $i(\mathrm{d}z) = \mathrm{d}^2 y = 0$ which implies by injectivity of i (exactness at E that $z \in Z(E)$. One verifies that the class of z in $H(E)$ only depends on $h \in H(G)$ and that, if one denotes if by ∂h, the corresponding mapping $\partial : H(G) \to H(E)$ satisfies the statement of proposition 1. ∂ is called the *connecting homomorphism.* ∎

An *exact couple* of vector spaces is an exact triangle of linear mapping a, b, c

$$\begin{tikzcd} F \arrow[rr, "b"] & & F \arrow[dl, "c"] \\ & E \arrow[ul, "a"] & \end{tikzcd}$$

involving two vector spaces E and F. In view of proposition 1, there is an easy way to produce exact couples, namely the exact triangles in homology associated with short exact sequences of differential spaces in which two of the three differential spaces involved coincide.

Given an exact couple as above, one constructs another one

$$\begin{tikzcd} F' \arrow[rr, "b'"] & & F' \arrow[dl, "c'"] \\ & E' \arrow[ul, "a'"] & \end{tikzcd}$$

called the *derived exact couple* by the following procedure. One takes $F' = b(F)$ and $b' = b \upharpoonright b(F) : F' \to F'$. From exactness at E, if follows that $\mathrm{d} = c \circ a$ is a differential on E (i.e. $\mathrm{d}^2 = 0$) and one defines E' to be the homology of the differential space $(E, \mathrm{d}) : E' = H(E, \mathrm{d})$. From exactness at F (the appropriate one) if follows that a maps $Z(E, \mathrm{d})$ into $b(F) = F'$ and $B(E, \mathrm{d})$ on $0 \in F$; a' is the induced mapping of $E' = Z(E, \mathrm{d})/B(E, \mathrm{d})$ in $F' = b(F)$. Again by exactness, c maps F into $Z(E, \mathrm{d})$ and the class of $c(f)$ in $E' = H(E, \mathrm{d})$ does only depend on $b(f) \in F'$, for $f \in F$; c' is the corresponding mapping of F' in E', one verifies that the triangle of linear mappings a', b', c' so defined is again exact.

By induction, one defines, for any integer $r \in \mathbb{N}$, the r^{th} *derived exact couple*

$$\begin{tikzcd} F_r \arrow[rr, "b_r"] & & F_r \arrow[dl, "c_r"] \\ & E_r \arrow[ul, "a_r"] & \end{tikzcd}$$

by $E_0 = E$, $F_0 = F$, $a_0 = a$, $b_0 = b$, $c_0 = c$ and $E_{r+1} = E'_r$, $F_{r+1} = F'_r$, $a_{r+1} = a'_r$, $b_{r+1} = b'_r$, $c_{r+1} = c'_r$. Setting $\mathrm{d}_r = c_r \circ a_r$, (E_r, d_r) is a differential space and E_{r+1} is its homology, $E_{r+1} = H(E_r, \mathrm{d}_r)$; $(E_r, \mathrm{d}_r)_{r \in \mathbb{N}}$ is *the spectral sequence* *associated to the exact couple*. Quite generally a sequence of differential spaces (V_r) such that $V_{r+1} = H(V_r)$ is called a *spectral sequence*.

Finally, if V is a differential space which is $\mathbb{Z}$-graded, $V = \underset{n \in \mathbb{Z}}{\oplus} V^n$ and if d is homogeneous of degree minus one, i.e. $\mathrm{d}\, V^n \subset V^{n-1}$ we call it a *chain complex*, the elements of $Z^n(V) = Z(V) \cap V^n$ will be called *n-cycles* and the one of $B^n(V) = B(V) \cap V^n$ will be called *n-boundaries*; in this case $H(V)$ is a graded vector space, with $H^n(V) = Z^n(V)/B^n(V)$ and the connecting homomorphism corresponding to a short exact sequence of chain complexes is also homogeneous of degree minus one. In the case where d is of degree one $V = \oplus V^n$ is called a *cochain complex*, the elements of V^n are called n-cochains, the ones of $Z^n(V)$ are called *n-cocycles* and the ones of $B^n(V)$ are called *n-coboundaries*; in this case $H(V)$ is called *the cohomology* of V, it is again a graded vector space and the connecting homomorphism associated to a short exact sequence of cochain complexes is homogeneous of degree one. A graded differential algebra is of course a cochain complex with these definitions. Notice also that if the differential space V is bigraded and if its differential is bi-homogeneous, then $H(V)$ is a bigraded vector space and that the connecting homomorphism of a short exact sequence of such bigraded differential spaces with differentials of fixed bidegree (r, s) is bihomogeneous of the same bidegree (r, s).

3.2. The exact couple relating the δ-cohomology modulo d and the δ-cohomology of the Weil - B.R.S. algebra

In this paragraph, $\mathfrak{g}$ is a fixed finite dimensional Lie algebra and the δ-cohomology $H^{*,*}(\mathrm{A}(\mathfrak{g}), \delta)$ of the Weil - B.R.S. algebra of $\mathfrak{g}$ will be denoted simply by $H(\delta)$. Thus, in view of theorem 10, $H^{2k,s}(\delta) = H^s(\mathfrak{g}, S^k \mathfrak{g}^*)$ and $H^{2k+1,s}(\delta) = 0$.

$\mathrm{d}\mathrm{A}(\mathfrak{g})$ is stable by δ and therefore δ induces a differential, again denoted by δ, on $\mathrm{A}(\mathfrak{g})/\mathrm{d}\mathrm{A}(\mathfrak{g})$ and the homology of $(\mathrm{A}(\mathfrak{g})/\mathrm{d}\mathrm{A}(\mathfrak{g}), \delta)$ is just the δ-cohomology modulo d that we want to compute; we denote it by $H(\delta, \mathrm{mod}\,(\mathrm{d}))$. Since $\mathrm{d}\mathrm{A}(\mathfrak{g})$ is a bigraded subspace of $\mathrm{A}(\mathfrak{g})$ and since δ is bihomogeneous (of bidegree $(0, 1)$), $H(\delta, \mathrm{mod}\,(\mathrm{d}))$ is a bigraded vector space; $H(\delta, \mathrm{mod}\,(\mathrm{d})) = \oplus H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d}))$ (with $H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d})) = 0$ whenever r or s is strictly negative).

By definition one has a short exact sequence of δ-differential spaces $0 \to \to \mathrm{d}\mathrm{A}(\mathfrak{g}) \xrightarrow{i} \mathrm{A}(\mathfrak{g}) \xrightarrow{p} \mathrm{A}(\mathfrak{g})/\mathrm{d}\mathrm{A}(\mathfrak{g}) \to 0$ from which one obtain exact sequences

$$\ldots \to H^{k,l}(\mathrm{d}\mathrm{A}(\mathfrak{g}), \delta) \xrightarrow{i^\#} H^{k,l}(\delta) \xrightarrow{p^\#} H^{k,l}(\delta, \mathrm{mod}\,(\mathrm{d})) \xrightarrow{\partial} H^{k,l+1}(\mathrm{d}\mathrm{A}(\mathfrak{g}), \delta) \xrightarrow{i^\#} \ldots.$$

On the other hand the linear mappings of $\mathrm{A}^{k,l}(\mathfrak{g})$ in $\mathrm{A}^{k+1,l}(\mathfrak{g})$ defined by $Q^{k,l} \mapsto (-1)^k \mathrm{d}Q^{k,l}$ induceds, for $k + l \geqslant 1$, isomorphisms of $\mathrm{A}^{k,l}(\mathfrak{g})/\mathrm{d}\mathrm{A}^{k-1,l}(\mathfrak{g})$ on $\mathrm{d}\mathrm{A}^{k,l}(\mathfrak{g}) = (\mathrm{d}\mathrm{A}(\mathfrak{g}))^{k+1,l}$ in view of therorem 9. These isomorphisms permute with δ, so one has, with the appropriate identifications,

$$H^{k,l}(\delta, \bmod (\mathrm{d})) = H^{k+1,l}(\mathrm{d}A(\mathfrak{g}), \delta) \text{ for } k+l \geqslant 1, \ (k, l \in \mathbb{N}).$$

So, by ckecking carefully what happens in small degrees, one obtains, (with the obvious identifications of $i^\#$ and ∂), the exact sequences for $r \geqslant 1$

$$\ldots \xrightarrow{i^\#} H^{r,s}(\delta) \xrightarrow{p^\#} H^{r,s}(\delta, \bmod (\mathrm{d})) \xrightarrow{\partial} H^{r-1,s+1}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\#} H^{r,s+1}(\delta) \xrightarrow{p^\#} \ldots$$

starting by $0 \to H^{1,0}(\delta) \xrightarrow{p^\#} H^{1,0}(\delta, \bmod (\mathrm{d})) \xrightarrow{\partial} \ldots$ for $r = 1$, and by $0 \to H^{r-1,0}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\#} H^{r,0}(\delta) \xrightarrow{p^\#} \ldots$ for $r \geqslant 2$. For $r = 0$ one has $H^{0,s}(\delta, \bmod (\mathrm{d})) \cong H^{0,s}(\delta)$ for any $s \in \mathbb{N}$, (induced by $p^\#$). Finally, by taking into account the structure of $H(\delta)$ given by theorem 10, one obtains the following result.

THEOREM 12. (a) *One has the following isomorphisms:*
$H^{0,s}(\delta, \bmod (\mathrm{d})) \cong H^s(\mathfrak{g})$, $\forall s \in \mathbb{N}$, *(induced by $p^\#$);*
$H^{2k+1,0}(\delta, \bmod (\mathrm{d})) \cong H^0(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) = \mathscr{T}_S^{k+1}(\mathfrak{g})$ and
$H^{2k+2,0}(\delta, \bmod (\mathrm{d})) = 0$, $\forall k \in \mathbb{N}$, *(induced by $i^\#$);*
$H^{2k+1,s}(\delta, \bmod (\mathrm{d})) \cong H^{2k,s+1}(\delta, \bmod (\mathrm{d}))$, $\forall k \in \mathbb{N}$ *and* $\forall s \in \mathbb{N}$, *(induced by ∂).*

(b) *For any $k \in \mathbb{N}$, one has the long exact sequence*

$$0 \to H^{2k+1,1}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\#} H^1(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) \xrightarrow{p^\#} H^{2k+2,1}(\delta, \bmod (\mathrm{d}))$$
$$\xrightarrow{\partial} H^{2k+1,2}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\#} \ldots \xrightarrow{i^\#} H^s(\mathfrak{g}, S^{k+1}\mathfrak{g}^*)$$
$$\xrightarrow{p^\#} H^{2k+2,s}(\delta, \bmod (\mathrm{d})) \xrightarrow{\partial} H^{2k+1,s+1}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\#} H^{s+1}(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) \xrightarrow{p^\#} \ldots$$

From the last isomorphisms of theorem 12-(a), it is clear that, in the exact sequence of theorem 12-(b), one may replace the $H^{2k+1,s+1}(\delta, \bmod (\mathrm{d}))$ by the $H^{2k,s+2}(\delta, \bmod (\mathrm{d}))$ for $s \in \mathbb{N}$, one then obtains for any $k \in \mathbb{N}$ the exact sequence

$$0 \to H^{2k,2}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\# \circ \partial^{-1}} H^1(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) \xrightarrow{p^\#} H^{2k+2,1}(\delta, \bmod (\mathrm{d}))$$
$$\xrightarrow{\partial^2} H^{2k,3}(\delta, \bmod (\mathrm{d})) \xrightarrow{i^\# \circ \partial^{-1}} \ldots \xrightarrow{i^\# \circ \partial^{-1}} H^s(\mathfrak{g}, S^{k+1}\mathfrak{g}^*)$$
$$\xrightarrow{p^\#} H^{2k+2,s}(\delta, \bmod (\mathrm{d})) \xrightarrow{\partial^2} H^{2k,s+2}(\delta, \bmod (\mathrm{d}))$$
$$\xrightarrow{i^\# \circ \partial^{-1}} H^{s+1}(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) \xrightarrow{p^\#} \ldots .$$

Thus, by introducing the $\mathbb{Z}^2$-graded spaces $H_+^{ev.,*}(\delta, \bmod (\mathrm{d}))$ and $H_+^*(\mathfrak{g}, S\mathfrak{g}^*)$ defined by $(H_+^{ev.,*}(\delta, \bmod (\mathrm{d})))^{r,s} = H^{2r,s}(\delta, \bmod (\mathrm{d}))$ and $(H_+^*(\mathfrak{g}, S\mathfrak{g}^*))^{r,s} = H^s(\mathfrak{g}, S^r\mathfrak{g}^*)$ for $r \geqslant 0$, $s \geqslant 0$ and $r+s \geqslant 1$ and by $(H_+^{ev.,*}(\delta, \bmod (\mathrm{d})))^{r,s} = (H_+^*(\mathfrak{g}, S\mathfrak{g}^*))^{r,s} = 0$ otherwise, one obtains *the exact couple* ϵ_0

$$\begin{tikzcd} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \arrow[rr, "\partial^2"] & & H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \arrow[dl, "i_0"] \\ & H_+^*(\mathfrak{g}, S\mathfrak{g}^*) \arrow[ul, "p_0"] & \end{tikzcd} \qquad (\epsilon_0)$$

where i_0 and p_0 denote the canonical mappings induced on theses spaces by $i^\# \circ \partial^{-1}$ and $p^\#$.

In view of the last isomorphism of theorem 12-(a), $H(\delta, \mathrm{mod}\,(\mathrm{d}))$ is completely known if we know $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$. The principle that we shall follow to compute $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$ will be to compute the spectral sequence $(E_r)_{r \in \mathbb{N}}$ associated to the exact couple ϵ_0, $(E_0 = H_+^*(\mathfrak{g}, S\mathfrak{g}^*))$, so the r^{th} derived exact couple ϵ_r reads

$$\begin{tikzcd} \partial^{2r} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \arrow[rr, "\partial^2"] & & \partial^{2r} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \arrow[dl, "i_r"] \\ & E_r \arrow[ul, "p_r"] & \end{tikzcd}$$

and we have (by exactness) isomorphisms

$$\partial^{2r} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong p_r(E_r) \oplus \partial^{2r+2} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})),$$

i.e. isomorphisms

$$H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong \bigoplus_{r=0}^{r=k} p_r(E_r) \ \oplus \partial^{2k+2} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$$

and therefore an isomorphism

$$H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong \bigoplus_r p_r(E_r)$$

since each element of $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$ has a finite bidegree and since ∂^2 decreases the second by 2. Thus knowing E_r's and the kernels of the p_r, one has $H(\delta, \mathrm{mod}\,(\mathrm{d}))$ up to an isomorphism; in fact it is sufficient to know the spectral sequence $(E_r, \mathrm{d}_r)_{r \in \mathbb{N}}$. We shall compute all that in the case of a reductive Lie algebra in the next paragraph.

We now want to give another useful description of the above operator ∂ which connects it with the «descending chain» equations [5].

LEMMA 3. (a) *Let $Q^{r,s} \in \mathrm{A}^{r,s}(\mathfrak{g})$ be such that there is a $Q^{r-1,s+1} \in \mathrm{A}^{r-1,s+1}(\mathfrak{g})$ such that $\delta Q^{r,s} + \mathrm{d} Q^{r-1,s+1} = 0$. Then there is a $Q^{r-2,s+2} \in \mathrm{A}^{r-2,s+2}(\mathfrak{g})$ such that $\delta Q^{r-1,s+1} + \mathrm{d} Q^{r-2,s+2} = 0$.*

(b) *Let $Q^{r,s} \in \mathrm{A}^{r,s}(\mathfrak{g})$ be such that there are $L^{r,s-1} \in \mathrm{A}^{r,s-1}(\mathfrak{g})$ and $L^{r-1,s} \in \mathrm{A}^{r-1,s}(\mathfrak{g})$ with $Q^{r,s} = \delta L^{r,s-1} + \mathrm{d} L^{r-1,s}$. Then $Q^{r,s}$ satisfies the assumption of*

(a) *and any* $Q^{r-1,s+1}$ *as in* (a) *is of the form* $Q^{r-1,s+1} = \delta L^{r-1,s} + \mathrm{d} L^{r-2,s+1}$ *for some* $L^{r-2,s+1} \in \mathrm{A}^{r-2,s+1}(\mathfrak{g})$.

Proof of (a). Apply δ to $\delta Q^{r,s} + \mathrm{d} Q^{r-1,s+1} = 0$; one obtains $\delta \mathrm{d} Q^{r-1,s+1} = = \mathrm{d}(-\delta Q^{r-1,s+1}) = 0$ so $\delta Q^{r-1,s+1} + \mathrm{d} Q^{r-2,s+2} = 0$ for some $Q^{r-2,s+2} \in \in \mathrm{A}^{r-2,s+2}(\mathfrak{g})$ in view of theorem 9 for d.

Proof of (b). One has $-\delta Q^{r,s} = \mathrm{d}\delta L^{r-1,s}$ so any $Q^{r-1,s+1}$ such that $\delta Q^{r,s} + + \mathrm{d} Q^{r-1,s+1} = 0$ satisfies $\mathrm{d}(Q^{r-1,s+1} - \delta L^{r-1,s}) = 0$. So, again by theorem 9, $Q^{r-1,s+1} = \delta L^{r-1,s} + \mathrm{d} L^{r-2,s-1}$.

The assumption (a) on $Q^{r,s}$ means that $Q^{r,s}$ defines a δ-cocycle modulo d and the statement of (a) is then that any $Q^{r-1,s+1}$ for which $\delta Q^{r,s} + \mathrm{d} Q^{r-1,s+1} = 0$ again defines a δ-cocycle modulo d. (b) means that if $Q^{r,s}$ defines a δ-coboundary modulo d then $Q^{r-1,s+1}$ also defines a coboundary modulo d. It follows that the class of $Q^{r-1,s+1}$ in $H^{r-1,s+1}(\delta, \mathrm{mod}\,(\mathrm{d}))$ is well defined and only depends on the class of $Q^{r,s}$ in $H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d}))$. We claim that the corresponding linear mapping in $H(\delta, \mathrm{mod}\,(\mathrm{d}))$ is just the above $\partial : H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d})) \to \to H^{r-1,s+1}(\delta, \mathrm{mod}\,(\mathrm{d}))$. A way to see it is to go back to the definition of the connecting homomorphism (see after proposition 1). Another way consists in checking directly the exactness of $H^{r,s}(\delta) \xrightarrow{p^{\#}} H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d})) \xrightarrow{\partial} H^{r-1,s+1}(\delta, \mathrm{mod}\,(\mathrm{d})) \xrightarrow{i^{\#}} H^{r,s+1}(\delta)$, where $p^{\#}$ is induced by the canonical projection $\mathrm{A}^{r,s}(\mathfrak{g}) \to \mathrm{A}^{r,s}(\mathfrak{g})/\mathrm{d}\mathrm{A}^{r-1,s}(\mathfrak{g})$, (a δ-cocycle $Q^{r,s}$ is canonically a δ-cocycle modulo d), and where $i^{\#}$ is induced by $(-1)^{r-1}\mathrm{d} : \mathrm{A}^{r-1,s}(\mathfrak{g}) \to \mathrm{A}^{r,s}(\mathfrak{g})$ considered as a mapping of $\mathrm{A}^{r-1,s}(\mathfrak{g})/\mathrm{d}\mathrm{A}^{r-2,s}(\mathfrak{g})$ in $\mathrm{A}^{r,s}(\mathfrak{g})$. ∎

LEMMA 4. *Let* $P \in \mathscr{T}_S^{k+1}(\mathfrak{g})$ *be an homogeneous invariant polynomial of degree* $k+1$ *on* $\mathfrak{g}$. *Then,* $P(F) = \mathbb{1} \otimes P \otimes \mathbb{1} \otimes \mathbb{1}$ *is an element of* $\mathrm{A}^{2k+2,0}(\mathfrak{g})$ *satisfying* $\mathrm{d}P(F) = \delta P(F) = 0$ *and there are* $Q^{2k+1-p,p} \in \mathscr{T}_\Lambda(\mathfrak{g})^{2k+1-p,p}$ *for* $0 \leqslant p \leqslant \leqslant 2k+1$ *such that* $P(F) = \mathrm{d} Q^{2k+1,0}$, $\delta Q^{2k+1-p,p} + \mathrm{d} Q^{2k-p,p+1} = 0$, $0 \leqslant p \leqslant 2k$ *and* $\delta Q^{0,2k+1} = 0$. *Furthermore one has* $Q^{0,2k+1} = \rho(P)(\chi) = \mathbb{1} \otimes \mathbb{1} \otimes \rho(P) \otimes \mathbb{1}$ *where* ρ *is the Cartan map* $\rho : \mathscr{T}_S^{k+1}(\mathfrak{g}) \to \mathscr{T}_\Lambda^{2k+1}(\mathfrak{g})$ (see in 1.4).

Firstly $\mathrm{d}P(F) = \delta P(F) = 0$ is obvious from the definitions and, furthermore, one has $i_X P(F) = 0$ and therefore $L_X P(F) = 0$ for any $X \in \mathfrak{g}$. Thus $P(F)$ is in fact in $\mathscr{T}_\Lambda(\mathfrak{g})$ and satisfies $(\mathrm{d} + \delta)P(F) = 0$, so by theorem 9, $P(F) = (\mathrm{d} + \delta)Q$ with $Q \in \mathscr{T}_\Lambda(\mathfrak{g})$ of total degree $2k+1$. Expanding Q in bidegree, $Q = = \sum_p Q^{2k+1-p,p}$, the $Q^{2k+1-p,p}$ satisfy the assumptions of the lemma. For the last point, one notices that an element of $\mathrm{A}^{0,n}(\mathfrak{g})$ is necessarily of the form $\omega(\chi) = \mathbb{1} \otimes \mathbb{1} \otimes \omega \otimes \mathbb{1}$ with $\omega \in \Lambda^n \mathfrak{g}^*$ and that $\omega(\chi)$ is invariant iff. ω is invariant, i.e. $\omega \in \mathscr{T}_\Lambda^n(\mathfrak{g})$; then the result just follows from the definition of the

Cartan map. ■

The $Q^{2k+1-p,p}$ of the last lemma are canonically δ-cocycles modulo d and one has, by the definition of ∂, $\partial[Q^{2k+1-p,p}] = [Q^{2k-p,p+1}]$ for the corresponding elements of $H(\delta, \mathrm{mod}\,(\mathrm{d}))$. Now if $Q' \in A(\mathfrak{g})$ is such that $P(F) = (\mathrm{d} + \delta)Q'$ then $Q' - Q = (\mathrm{d} + \delta)L$ so $[Q'^{2k+1-p,p}] = [Q^{2k+1-p,p}]$ and therefore $P \mapsto$ $\mapsto [Q^{2k+1-p,p}]$ is a well defined linear mapping $j^{k,p+1} : \mathscr{T}_S^{k+1}(\mathfrak{g}) \to H^{2k+1-p,p}$ $(\delta, \mathrm{mod}\,(\mathrm{d}))$ for $k \in \mathbb{N}$ and $0 \leqslant p \leqslant 2k+1$. One has $j^{k,p+1} = \partial \circ j^{k,p}$, which makes contact between the operator ∂ and the «descending chain». *As far as one is only interested in anomalies in even dimension* $2k$, i.e. on $H^{2k,1}(\delta, \mathrm{mod}\,(\mathrm{d}))$, *one notices that* $j^{k,1} : \mathscr{T}_S^{k+1}(\mathfrak{g}) \to H^{2k,1}(\delta, \mathrm{mod}\,(\mathrm{d}))$ *is an isomorphism* since $H^{2k+1,1}(\delta) = H^{2k+1,0}(\delta) = 0$; thus all $H^{2k,1}(\delta, \mathrm{mod}\,(\mathrm{d}))$ is obtained by the construction described in [5].

3.3. The case of a reductive Lie algebra

In the following $\mathfrak{g}$ will be a finite dimensional reductive Lie algebra and $P = \bigoplus_{k \geqslant 0} P^{2k+1}$ will denote the space of primitive forms on $\mathfrak{g}$, P^{2k+1} being the space of primitive forms of degree $2k+1$, (see in 1.2-c). We choose once for all a transgression $\tau : P \to \mathscr{T}_S(\mathfrak{g})$ so $\mathscr{T}_S(\mathfrak{g}) = S\tau(P)$ and $H^*(\mathfrak{g}, S\,\mathfrak{g}^*) = S\tau(P) \otimes \Lambda P$ (see in 1.4) is a graded commutative algebra if one defines the degree by $\mathbf{1} \otimes \Lambda^r P^{2k+1} \subset (S\tau(P) \otimes \Lambda P)^{r(2k+1)}$ and $S^r\tau(P^{2k+1}) \otimes \mathbf{1} \subset (S\tau(P) \otimes \Lambda P)^{r(2k+2)}$. One has $\dim(P) = \Sigma \dim(P^{2k+1}) = \mathrm{rank}\,(\mathfrak{g})$ therefore there is an integer $r_M(\mathfrak{g})$, such that $\dim(P^{2r_M(\mathfrak{g})+1}) \geqslant 1$ and $\dim(P^{2k+1}) = 0$ for $k > r_M(\mathfrak{g})$ so $P = \bigoplus_{k=0}^{k=r_M(\mathfrak{g})} P^{2k+1}$.

Let us introduce the subspace $P_r = \bigoplus_{k \geqslant r} P^{2k+1}$ of $P(= P_0)$ and define the subalgebras $\mathscr{T}_r$ and $E_r = \mathscr{T}_r^+$ of $\mathscr{T}_S(\mathfrak{g}) \otimes \mathscr{T}_\Lambda(\mathfrak{g})$, $(r \in \mathbb{N})$, by $\mathscr{T}_r = (S\tau(P_r)) \otimes (\Lambda P_r)$ and $E_r = \bigoplus_{m+n \geqslant 1} (S^m\tau(P_r)) \otimes (\Lambda^n P_r)\ (= \mathscr{T}_r^+)$. We have:

$$E_r = E_{r+1} \oplus \left(\bigoplus_{m+n \geqslant 1} (S^m\tau(P^{2r+1})) \otimes (\Lambda^n P^{2r+1})\right) \otimes \mathscr{T}_{r+1} = E_{r+1} \oplus E_r^r$$

(i.e. E_r^r involves at least one primitive element of degree $2r+1$). The identification $H^*(\mathfrak{g}, S\,\mathfrak{g}^*) = \mathscr{T}_S(\mathfrak{g}) \otimes \mathscr{T}_\Lambda(\mathfrak{g}) = \mathscr{T}_0$ leads to $H_+^*(\mathfrak{g}, S\,\mathfrak{g}^*) = E_0$; furthermore we have $E_r = 0$ for $r > r_M(\mathfrak{g})$. Let d_r be the unique antiderivation of $\mathscr{T}_r$ such that $\mathrm{d}_r(\mathbf{1} \otimes P_{r+1}) = 0$, $\mathrm{d}_r(\tau(P_r) \otimes \mathbf{1}) = 0$ and $\mathrm{d}_r(\mathbf{1} \otimes \alpha) = \tau(\alpha) \otimes \mathbf{1}$ for $\alpha \in P^{2r+1}$. Then we have $\mathrm{d}_r^2 = 0$, $\mathrm{d}_r(E_r) \subset E_r$ so the homology $H(E_r, \mathrm{d}_r)$ is well defined.

LEMMA 5. *We have* $E_{r+1} = H(E_r, \mathrm{d}_r)$ *for* $r \in \mathbb{N}$, *i.e. the sequence* $(E_r, \mathrm{d}_r)_{r \in \mathbb{N}}$ *is a spectral sequence.*

Proof. We have $d_r(E_{r+1}) = 0$ and $d_r(E_r^r) \subset E_r^r$ so all what we have to prove is that the homology $H(E_r^r, d_r)$ vanishes. Define d_r' to be the unique antiderivation of $\mathscr{T}_r$ such that $d_r'(\mathbb{1} \otimes P_r) = 0$, $d_r'(\tau(P_{r+1}) \otimes \mathbb{1}) = 0$ and $d_r'(\tau(\alpha) \otimes \mathbb{1}) = \mathbb{1} \otimes \alpha$ for $\alpha \in P^{2r+1}$. Then $d_r'(E_r^r) \subset E_r^r$ and the derivation $d_r d_r' + d_r' d_r$ coincides on $(S^m \tau(P^{2r+1})) \otimes (\Lambda^n P^{2r+1}) \otimes \mathscr{T}_{r+1}$ with the multiplication by the number $n + m$ so, since $E_r^r = (\oplus_{n+m \geqslant 1} (S^m \tau(P^{2r+1})) \otimes (\Lambda^n P^{2r+1})) \otimes \mathscr{T}_{r+1}$, any d_r-closed element of E_r^r is d_r-exact, which achieves the proof. ∎

We shall show that $(E_r, d_r)_{r \in \mathbb{N}}$ is the spectral sequence associated to the exact couple ϵ_0, and it is why we use this notation; for that, we need the following lemma.

LEMMA 6. (Generalised «transgression» lemma). *Let $X \in E_r$, then, there are $Q_k \in A(\mathfrak{g})$ for $k = 1, 2, \dots, 2r + 2$ such that we have: $dX + \delta Q_1 = 0$, $dQ_k + \delta Q_{k+1} = 0$, for $1 \leqslant k \leqslant 2r$, and $dQ_{2r+1} + \delta Q_{2r+2} = d_r X$. In other words, there is an element α of $H_+^{ev,*}(\delta, \mathrm{mod}\,(d))$ such that $d_r X = i_0(\alpha)$ $(= i \circ \partial^{-1}(\alpha))$ and $\partial^{2r}\alpha = p_0(X)$ $(= p^{\#}(X))$; (Take α to be the class of Q_{2r} as above in $H_+^{ev,*}(\delta$, mod (d))).*

In the first part of this lemma, $\mathscr{T}_S(\mathfrak{g}) \otimes \mathscr{T}_\Lambda(\mathfrak{g}) = S\tau(P) \otimes \Lambda P$ is identified with a subalgebra of $A(\mathfrak{g})$ (in fact of $\mathbb{1} \otimes S\mathfrak{g}^* \otimes \Lambda \mathfrak{g}^* \otimes \mathbb{1}$) by writing $\tau(\omega) = \tau(\omega)(F) = \mathbb{1} \otimes \tau(\omega) \otimes \mathbb{1} \otimes \mathbb{1} \in A(\mathfrak{g})$ and $\omega = \omega(\chi) = \mathbb{1} \otimes \mathbb{1} \otimes \omega \otimes \mathbb{1} \in A(\mathfrak{g})$ for $\omega \in P$; elements of $S\tau(P) \otimes \Lambda P$ are thus identified with the corresponding δ-cocycles.

Proof. It is sufficient to consider monomials

$$X = \prod_i \tau(\omega_i') \otimes \omega_0 \dots \omega_n = \prod_i \tau(\omega_i')(F)\,\omega_0(\chi) \dots \omega_n(\chi)$$

where the ω' and the ω are homogeneous primitive forms of degrees greater then or equal to $2r + 1$. Introducing, (as in the proof of lemma 4), for each $0 \leqslant p \leqslant n$, $L_p \in \mathscr{T}_\Lambda(\mathfrak{g})$ such that $\tau(\omega_p)(F) = (d + \delta) L_p$ one has $L_p = \sum_s L_p^{2r_p+1-s,s}$ with $L_p^{0,2r_p+1} = \omega_p(\chi)$, $2r_p + 1 = $ degree (ω_p), (see Lemma 4), and

$$(d + \delta) \prod_i \tau(\omega_i')(F) L_0 \dots L_n = \sum_{p=0}^{p=n} (-1)^p \prod_i \tau(\omega_i')(F)\tau(\omega_p)(F) L_0 \dots \hat{L}_p \dots L_n .$$

Expanding the last equation in degreasing «δ-degree» (i.e. in the second partial degree of the bidegree) yields a sequence of equations. The first $2r + 3$ give

$\delta X = 0$ and the equations of the lemma with explicit Q_k. In particular the $(2r+3)^{\text{th}}$ equation reads

$$\mathrm{d}Q_{2r+1} + \delta Q_{2r+2} =$$

$$= \sum_{\{p \text{ such that } \omega_p \in P^{2r+1}\}} (-1)^p \prod_i \tau(\omega_i')(F)\,\tau(\omega_p)(F)\,\omega_0(\chi)\ldots\hat{\omega}_p(\chi)\ldots\omega_n(\chi) =$$

$$= \mathrm{d}_r X. \qquad \blacksquare$$

It is worth noticing here that this proof can be adapted to give an easy proof of theorem 6. Notice also that if $\omega' \in P^{2s+1}$ with $s < r$, we may do the same thing for $\tau(\omega')X$ as we did for X, but now one has $p_0(\tau(\omega)X) = 0$, i.e. $\tau(\omega')(F)\,X = \mathrm{d}\alpha + \delta\beta$ for some α and β in $A(\mathfrak{g})$.

We are now ready to identify $(E_r, \mathrm{d}_r)_{r \in \mathbb{N}}$ with the spectral sequence associated to ϵ_0.

THEOREM 13. *For any $r \in \mathbb{N}$, one has an exact triangle ϵ_r*

$$\begin{tikzcd} \partial^{2r} H_+^{ev,*}(\delta, \bmod (\mathrm{d})) \arrow[rr, "\partial^2"] & & \partial^{2r} H_+^{ev,*}(\delta, \bmod (\mathrm{d})) \arrow[dl, "i_r"] \\ & E_r \arrow[ul, "p_r"] & \end{tikzcd}$$

where p_r is induced, by restriction to $E_r \subset E_0$, by *p_0 and where $i_r(\partial^{2r}\alpha)$ is the component of $i_0(\alpha)$ on E_r in the direct sum decomposition $E_0 = E_r \oplus (\bigoplus_{s=0}^{s=r-1} E_s^s)$. One has furthermore $\mathrm{d}_r = i_r \circ p_r$, so ϵ_r identifies with the r^{th} derived exact couple of ϵ_0 and $(E_r, \mathrm{d}_r)_{r \in \mathbb{N}}$ is the associated spectral sequence.*

Given $X \in E_r$, let α be as in lemma 6, i.e. $\partial^{2r}\alpha = p_0(X) = p_r(X)$ and $\mathrm{d}_r(X) = i_0(\alpha)$; one has:

$$i_r \circ p_r(X) = i_r(\partial^{2r}\alpha) = \mathrm{proj}_{E_r} i_0(\alpha) = \mathrm{proj}_{E_r} \mathrm{d}_r X = \mathrm{d}_r X;$$

thus $\mathrm{d}_r = i_r \circ p_r$. By lemma 5, we have $E_{r+1} = H(E_r, \mathrm{d}_r)$ so, by induction on r, ϵ_r identifies with the r^{th} derived exact couple of ϵ_0. $\blacksquare$

It follows from the results of 3.2 that one has an isomorphism

$$H_+^{ev,*}(\delta, \bmod (\mathrm{d})) \cong \bigoplus_r p_r(E_r) = \bigoplus_{r=0}^{r=r_M(\mathfrak{g})} p^{\#}(E_r).$$

Furthermore, from $E_r = 0$ for $r > r_M(\mathfrak{g})$ it follows that $\partial^2 : \partial^{2r} H_+^{ev,*}(\delta, \bmod (\mathrm{d})) \to \partial^{2r} H_+^{ev,*}(\delta, \bmod (\mathrm{d}))$ is an isomorphism whenever $r > r_M(\mathfrak{g})$; therefore

$\partial^{2r} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong \partial^{2(r+k)} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$ for any $r > r_M(\mathfrak{g})$ and any $k \geqslant 0$ which implies that *we have* $\partial^{2r} H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) = 0$ for $r > r_M(\mathfrak{g})$ since any $x \in H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$ is such that there is an integer $n \geqslant 0$ for which $\partial^{2n} x = 0$.

It remains to compute $\ker(p_r)$. One has the following result.

PROPOSITION 2. *One has* $\ker(p_r) = \bigoplus_{k=r}^{k=r_M(\mathfrak{g})} \mathrm{Im}\,(\mathrm{d}_k)$.

Let $X \in E_r$ be such that $p_r(X) = 0$; then $\mathrm{d}_r X = 0$. So $X = \mathrm{d}_r Y + Z$ with $Z \in E_{r+1} = H(E_r, \mathrm{d}_r)$. One has then $p_{r+1}(Z) = p_r(Z) = p_r(X) - p_r \mathrm{d}_r Y = 0$, so $X \in \mathrm{Im}\,(\mathrm{d}_r) \oplus \ker(p_{r+1})$ and the statement of proposition 2 follows by induction. ■

As a consequence of the last results one has the following isomorphisms

$$p_r(E_r) \cong E_r/\ker(p_r) = \bigoplus_{s=r}^{s=r_M(\mathfrak{g})} (N_s/\mathrm{d}_s N_s) \otimes \mathscr{T}_{s+1} \cong$$
$$\cong \bigoplus_{s=r}^{s=r_M(\mathfrak{g})} (\mathrm{d}_s N_s) \otimes \mathscr{T}_{s+1}$$

with $N_s = \bigoplus_{m+n \geqslant 1} S^m \tau(P^{2s+1}) \otimes \Lambda^n P^{2s+1}$.

These formulae end the computation of $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$ and therefore the computation of $H(\delta, \mathrm{mod}\,(\mathrm{d}))$ in the case of a reductive $\mathfrak{g}$; in particular the dimensions of the $H^{r,s}(\delta, \mathrm{mod}\,(\mathrm{d}))$ follow from the dimensions of the space P^{2k+1}, (see for instance in [2] for such computations).

Practically the isomorphism $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong \oplus\, p_r(E_r)$ may be realized by the following procedure. Given a homogeneous basis (ω_p) of P, choose invariant L_p, (as in the proof of lemma 6), such that $\tau(\omega_p)(F) = (\mathrm{d} + \delta) L_p$; then the construction in the proof of lemma 6 gives a linear mapping $\psi : \bigoplus_{r=0}^{r=r_M(\mathfrak{g})} E_r \to H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d}))$. Choose, for each r, a supplementary K_r in E_r to $\ker(p_r)$; ψ restricted to $\oplus K_r$ gives, when combined with $\oplus K_r \cong \oplus\, p_r(E_r)$, a realization of the isomorphism $H_+^{ev,*}(\delta, \mathrm{mod}\,(\mathrm{d})) \cong \oplus\, p_r(E_r)$ by (independent) representative δ-cocycles modulo d.

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